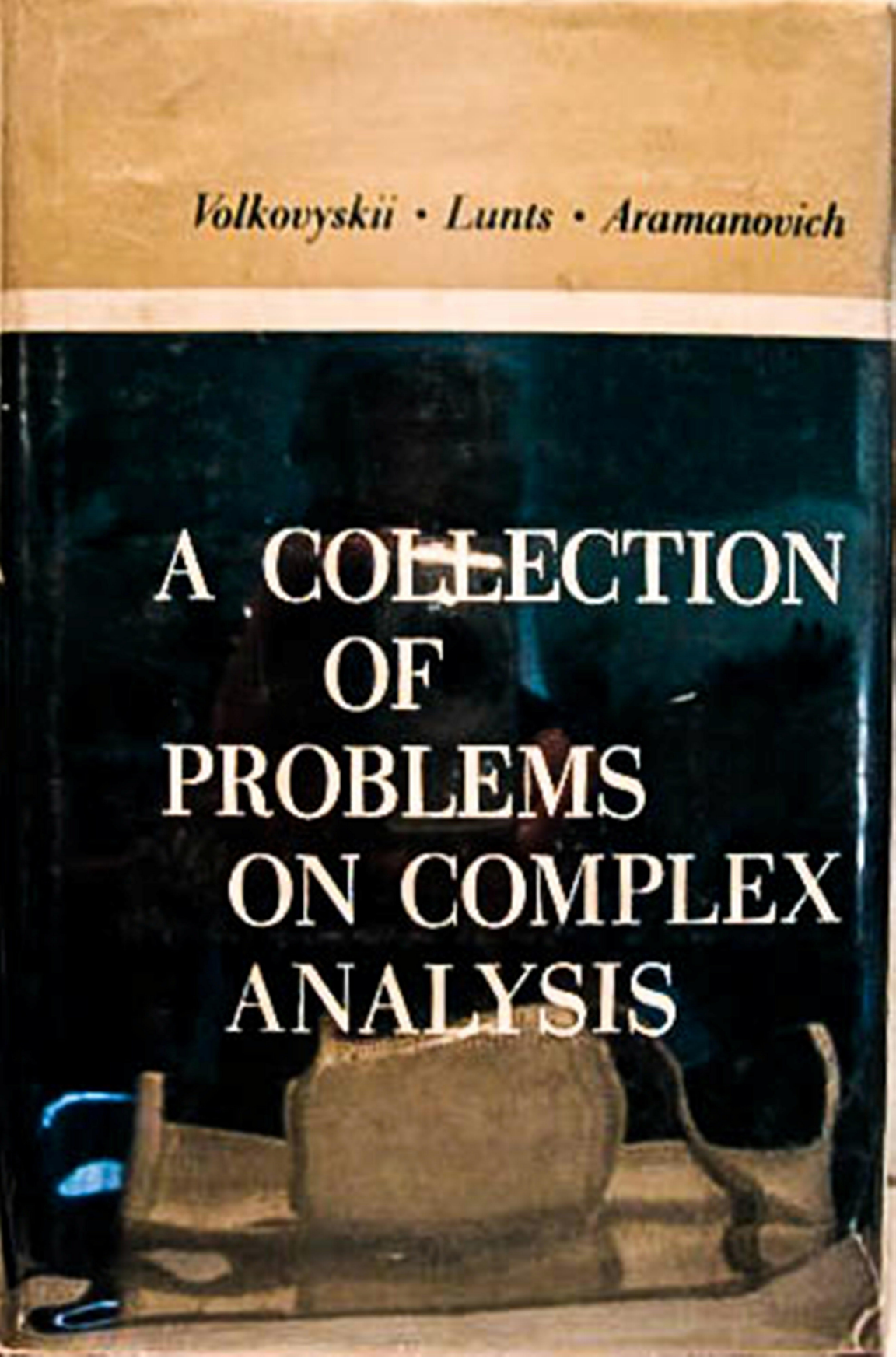
Volkovyskiĭ • Lunts • Aramanovich

A COLLECTION OF PROBLEMS ON COMPLEX ANALYSIS



A COLLECTION OF PROBLEMS ON COMPLEX ANALYSIS

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A Collection of Problems on COMPLEX ANALYSIS

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This book is an edited translation of the original Russian *Сборник задач по теории функций комплексного переменного* (*Sbornik zadach po teoriyi funktsii kompleksnogo peremennogo*), published in 1960 by Fizmatgiz, Moscow

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FOREWORD

A Collection of Problems on Complex Analysis is intended mainly for students of the physico-mathematical and mechanico-mathematical faculties of universities and also for students of the physico-mathematical faculties of teachers training colleges.

The authors believe also that the collection will be useful for those specialising in the mechanics of continuous media (hydrodynamics, the theory of elasticity) and electrical engineering, since it contains a large number of problems either on the direct application of the theory of functions of a complex variable to the given subject, or on questions forming their mathematical basis (conformal mappings, harmonic functions, potentials, integrals of Cauchy type, etc.).

Problems connected with the compulsory university course on the theory of functions of a complex variable are arranged mainly in Chapters I, II (§ 1, 3 and 4), IV, V, VII, X (§ 1).

The book also contains groups of problems which go beyond the range of the syllabuses. Some of these may be used as the basis of course work and as material for study in seminars on the theory of functions of a complex variable.

It seems to us that the book deals with the main sections of the theory of functions of a complex variable approximately in accordance with the curriculum.

For the convenience of those using the book, in the contents besides the titles of the chapters and sections, the main groups of problems contained in them have sometimes been enumerated (this refers mainly to the basic course material).

It is assumed that the users of the collection are acquainted with the corresponding sections of the course in the theory of functions of a complex variable contained in the book by A. I. Markushevich *A Short Course on the Theory of Analytic Functions*, and therefore as a rule references are not given to this textbook. If additional material is used the necessary information of the source and also references to the literature are given. The following notation is used for the books most frequently cited:

[1] A. I. Markushevich, *Theory of Analytic Functions* (*Teoriya analiticheskikh funktsii*), Gostekhizdat, Moscow, 1950.

[2] A. I. Markushevich, *Short Course of the Theory of Analytic Functions* (*Kratkii kurs teorii analiticheskikh funktsii*), Gostekhizdat, Moscow, 1957.

[3] M. A. Lavrent'yev and B. V. Shabat, *Methods of the Theory of Functions of a Complex Variable* (*Metody teorii funktsii kompleksnogo peremennogo*), 2nd ed. Fizmatgiz, Moscow, 1958.

[4] I. I. Privalov, *Introduction to the Theory of Functions of a Complex Variable* (*Vvedenie v teoriyu funktsii kompleksnogo peremennogo*), 10th ed. Fizmatgiz, Moscow, 1960.

All the hints for the solution of the problems are given in the main text. The most difficult problems, the numbers of which are marked by asterisks are provided with solutions located in the answers.

In compiling the collection use was made of textbooks and monographs by both Russian and foreign authors.

The manuscript of the book was reviewed in the Faculty of the Theory of Functions at Moscow University and the authors express their sincere gratitude to the members of the faculty and especially to B. V. Shabat for much valuable advice.

Much help in the checking and arrangement of some of the problems was given by the pupils of Professors L. I. Volkovyskii. B. A. Vertgeim, S. Ya. Gusman, V. V. Dumkin, V. G. Mikhal'chuk, Yu. L. Rodin, E. V. Starkova and I. I. Filimonova. I. I. Pesin took part in the choice of the problems of § 3, Chapter IX. A number of problems were proposed by L. E. El'sgol'ts. Extremely important work was done on the collection by the editor of the book N. A. Ugarova.

The authors express their deep obligation to all those mentioned.

THE AUTHORS

CHAPTER I

COMPLEX NUMBERS AND FUNCTIONS OF A COMPLEX VARIABLE

IN THIS chapter and generally everywhere in the book where nothing is said to the contrary the following notations are used: $z = x+iy = re^{i\phi}$, $w = u+iv = \varrho e^{i\theta}$ (x, y, u, v, r, ϱ, φ and θ are real numbers, $r \geqslant 0$, $\varrho \geqslant 0$); $\operatorname{Re} z = x$, $\operatorname{Im} z = y$, $\arg z = \varphi$, $|z| = r$, $\overline{z} = x-iy$. In the absence of any additional indications the principal value of the argument $\arg z$ is defined by the inequalities $-\pi < \arg z \leqslant \pi$; the complex plane whose points are represented by the complex number z, will be called the z-plane; usually the terms "complex number z" and "point z" are used as synonyms.

§ 1. Complex numbers

1. Perform the operations indicated:

(1) $1/i$;

(2) $\dfrac{1-i}{1+i}$;

(3) $\dfrac{2}{1-3i}$;

(4) $(1+i\sqrt{3})^3$.

2. Find the modulus and argument of each of the following complex numbers (a and b are real numbers):

(1) $3i$;
(2) -2;
(3) $1+i$;
(4) $-1-i$;
(5) $2+5i$;
(6) $2-5i$;
(7) $-2+5i$;
(8) $-2-5i$;
(9) bi ($b \neq 0$);
(10) $a+bi$ ($a \neq 0$).

3. Find all the values of the following roots and plot them:

(1) $\sqrt[3]{1}$;
(2) $\sqrt[3]{i}$;
(3) $\sqrt[4]{(-1)}$;
(4) $\sqrt[6]{(-8)}$;
(5) $\sqrt[8]{1}$;
(6) $\sqrt{(1-i)}$;
(7) $\sqrt{(3+4i)}$;
(8) $\sqrt[3]{(-2+2i)}$;
(9) $\sqrt[5]{(-4+3i)}$,

4. Prove that both values of $\sqrt{(z^2-1)}$ lie on the straight line passing through the coordinate origin and parallel to the bisector of the

internal angle of the triangle, with vertices at the points -1, 1 and z, which passes through the vertex z.

5. Let m and n be integers. Prove that $(\sqrt[n]{z})^m$ has $n/(n, m)$ different values, where (n, m) is the greatest common divisor of the numbers m and n. Verify that the sets of values of $(\sqrt[n]{z})^m$ and $\sqrt[n]{(z^m)}$ are identical if and only if $(n, m) = 1$, that is, if n and m are coprimes.

---

6. Prove the following inequalities from geometrical considerations:

(1) $|z_1+z_2| \leqslant |z_1| + |z_2|$;

(2) $|z_1-z_2| \geqslant \big||z_1|-|z_2|\big|$;

(3) $|z-1| \leqslant \big||z|-1\big|+|z||\arg z|$.

In each case explain when the sign of equality applies.

7. Prove the identity

$$|z_1+z_2|^2+|z_1-z_2|^2 = 2(|z_1|^2+|z_2|^2)$$

and explain its geometrical meaning.

8. Prove that if $|z_1| = |z_2| = |z_3|$, then

$$\arg\frac{z_3-z_2}{z_3-z_1} = \frac{1}{2}\arg\frac{z_2}{z_1}.$$

9. Prove that if $z_1+z_2+z_3 = 0$ and $|z_1| = |z_2| = |z_3| = 1$, then the points z_1, z_2, z_3 are the vertices of an equilateral triangle inscribed in the unit circle.

10. Prove that if $z_1+z_2+z_3+z_4 = 0$ and $|z_1| = |z_2| = |z_3| = |z_4|$, then the points z_1, z_2, z_3, z_4 are either the vertices of a rectangle or pairs of them are identical.

11. Find the vertices of the regular n-gon if its centre is at the point $z = 0$, and one of its vertices z_1 is known.

12. The points z_1 and z_2 are adjacent vertices of a regular n-gon. Find the vertex z_3, adjacent to z_2 ($z_3 \neq z_1$).

13. Given three vertices of a parallelogram z_1, z_2, z_3, find the fourth vertex z_4, opposite to the vertex z_2.

14. Under what conditions do three points z_1, z_2, z_3, no two of which are coincident, lie on a straight line?

15. Under what conditions do four points z_1, z_2, z_3, z_4, no two of which are coincident, lie on a circle or straight line?

16*. The points $z_1, z_2, \ldots, z_n$ lie on one side of a certain straight line passing through the coordinate origin. Prove that the points $1/z_1, 1/z_2, \ldots, 1/z_n$ possess a similar property (with respect to what straight line?) and that $z_1+z_2+\ldots+z_n \neq 0$; $1/z_1+1/z_2+\ldots+1/z_n \neq 0$.

17. Prove that if $z_1+z_2+\ldots+z_n=0$, then any straight line passing through the coordinate origin divides the points $z_1, z_2, \ldots, z_n$, provided only that these points do not lie on this straight line.

18. Prove that any straight line passing through the centre of gravity of the material points $z_1, z_2, \ldots, z_n$ with masses $m_1, m_2, \ldots, m_n$, divides these points provided only that they do not lie on this straight line.

Explain the geometrical meaning of the relations given in problems **19–29.**

19. $|z-z_0| < R$; $|z-z_0| > R$; $|z-z_0| = R$.

20. $|z-2|+|z+2| = 5$.

21. $|z-2|-|z+2| > 3$.

22. $|z-z_1| = |z-z_2|$.

23. (1) $\operatorname{Re} z \geqslant C$; (2) $\operatorname{Im} z < C$.

24. $0 < \operatorname{Re}(iz) < 1$.

25. $\alpha < \arg z < \beta$; $\alpha < \arg(z-z_0) < \beta$ $(-\pi < \alpha < \beta \leqslant \pi)$.

26. $|z| = \operatorname{Re} z+1$.

27. $\operatorname{Re} z+\operatorname{Im} z < 1$.

28. $\operatorname{Im}\dfrac{z-z_1}{z-z_2} = 0$; $\operatorname{Re}\dfrac{z-z_1}{z-z_2} = 0$.

29. (1) $|z| < \arg z$, if $0 \leqslant \arg z < 2\pi$;
(2) $|z| < \arg z$, if $0 < \arg z \leqslant 2\pi$.

In problems **30–33** it is required to determine the families of curves in the z-plane defined by the corresponding equations.

30. (1) $\operatorname{Re}\dfrac{1}{z} = C$; (2) $\operatorname{Im}\dfrac{1}{z} = C$ $(-\infty < C < \infty)$.

31. (1) $\operatorname{Re} z^2 = C$; (2) $\operatorname{Im} z^2 = C$ $(-\infty < C < \infty)$.

32. $\left|\dfrac{z-z_1}{z-z_2}\right| = \lambda$ $(\lambda > 0)$.

33. $\arg\dfrac{z-z_1}{z-z_2} = \alpha$ $(-\pi < \alpha \leqslant \pi)$.

34. (1) A family of curves in the z-plane is defined by the equation

$$|z^2-1| = \lambda \quad (\lambda > 0).$$

For what values of λ do the curves of the family consist of a simple curve and for what values do they split into two branches?

(2) Answer the same questions for the family

$$|z^2+az+b| = \lambda \quad (\lambda > 0).$$

---

35. The function $\arg z$ is defined uniquely at all points $z \neq 0$, if we put $|z|-2\pi < \arg z \leqslant |z|$. What is the geometrical locus of the points at which the continuity of the function $\arg z$ defined in this way breaks down?

36. What is the geometrical locus of the points at which the continuity of the function $\arg z$, uniquely defined for any $z \neq 0$ by the inequalities

$$\log|z|-2\pi < \arg z \leqslant \log|z|$$

breaks down?

37. For $z = 2$ the initial value of $\arg f(z)$ is taken to be equal to 0. The point z makes a complete rotation anticlockwise along a circle with centre at the coordinate origin and returns to the point $z = 2$. Considering that $\arg f(z)$ varies continuously during the motion of the point z, find the value of $\arg f(2)$ after the given rotation, if:

(1) $f(z) = \sqrt{(z-1)}$;

(2) $f(z) = \sqrt[3]{(z-1)}$;

(3) $f(z) = \sqrt{(z^2-1)}$;

(4) $f(z) = \sqrt{(z^2+2z-3)}$;

(5) $f(z) = \sqrt{\left(\frac{z-1}{z+1}\right)}$.

Stereographic projection

38. Derive the formulae of stereographic projection expressing the coordinates (ξ, η, ζ) of a point P of the sphere of diameter 1, touching the z-plane at the coordinate origin in terms of the coordinates (x, y) of the corresponding point z. Also express x and y in terms of ξ, η, ζ (the axes of ξ and η are assumed to coincide with the axes of x and y respectively).

REMARK. In problem 38 a correspondence is established between the points of the complex plane and the points of a sphere of radius 1/2 touching this plane. Another correspondence exists in which the sphere is of radius 1 and the z-plane passes through its centre. See, for example, [1, Chapter I, § 5].

39. What are the images on the sphere of the points $1, -1, i, \frac{1-i}{\sqrt{2}}$?

40. What is the image on the plane of the parallel with latitude $\beta\left(-\frac{\pi}{2}<\beta<\frac{\pi}{2}\right)$? What corresponds to the north and south poles?

41. Find on the sphere the images:

(1) Of the rays $\arg z = \alpha$;

(2) Of the circles $|z| = r$.

42. What is the position on the sphere of pairs of points symmetrical:

(1) With respect to the point $z = 0$;

(2) With respect to the real axis;

(3) With respect to the unit circle?

43. Subject to what conditions are the points z_1, and z_2 the stereographic projections of two diametrically opposite points of the sphere?

44. Which points of the sphere correspond to the points z and $\frac{1}{z}$?

45. Find on the sphere the images of the domains defined by the inequalities:

(1) $\operatorname{Im} z > 0$;

(2) $\operatorname{Im} z < 0$;

(3) $\operatorname{Re} z > 0$;

(4) $\operatorname{Re} z < 0$;

(5) $|z| < 1$;

(6) $|z| > 1$.

46. What corresponds on the sphere to a family of parallel straight lines on the plane?

47. Prove that in stereographic projection, circles on the sphere project into circles or straight lines on the plane. Which circles on the sphere correspond to straight lines?

48. Let K be a circle on the plane corresponding to the circle K' on the sphere, let N be the north pole of the sphere, and let S be the vertex of the cone touching the sphere along K' (it is assumed that K' is not a great circle). Prove that the centre of the circle K lies on the ray NS. Consider the case when K' is a great circle.

49. Prove that in stereographic projection, angles between curves on the sphere are equal to the angles between their images on the plane. Prove also that the sense of angles is preserved if on the sphere the positive direction of rotation is defined with respect to the inward normal.

50. Find the length $k(z, a)$ of the chord connecting points of the sphere which correspond to the points z and a. Consider also the case when a is the point at infinity.

51. Given two points z_1 and z_2 (one of them may be at infinity), find the geometrical locus of the points of the z-plane which correspond to a circle on the sphere equidistant from the images of the given points.

Quaternions

Quaternions† are numbers which are linear combinations with real coefficients of four units—one being real and three imaginary i, j, k;

$$q = a+bi+cj+dk\,,$$

the multiplication of units being defined by the equations

$$i^2 = j^2 = k^2 = -1;$$

$$ij = -ji = k,\ jk = -kj = i,\quad ki = -ik = j;$$

$$1.i = i.1 = i,\quad 1.j = j.1 = j,\quad 1.k = k.1 = k.$$

Quaternions with identical components (coefficients of the units) are considered to be equal. In the addition of quaternions the corresponding units are added. Multiplication is subject to the associative and distributive laws. Thus, for example,

$$(2i+4j)(3j-2k) = (2i)(3j)+(4j)(3j)+(2i)(-2k)+(4j)(-2k)$$

$$= 2.3.i.j+4.3.j^2+2(-2).i.k+4.(-2).j.k = 6k-12+4j-8i\,.$$

The numbers $q = a+bi+cj+dk$ and $\bar{q} = a-bi-cj-dk$ are said to be *conjugate* to one another. The number $N(q) = a^2+b^2+c^2+d^2$ is called the *norm* of the quaternion q.

52. Prove that in the quaternion system the operation of subtraction is defined uniquely and that for divisors different from zero the operation of division is defined uniquely.

53. Prove that $\overline{q_1}\overline{q_2} = \overline{q}_1 \cdot \overline{q}_2$.

54. Prove the equality

$$N(q_1q_2) = N(q_1)\,N(q_2)$$

and using this equality show that the product of two integers each of which can be represented as the sum of the squares of four integers is also a number which can be represented as the sum of four squares. Obtain this representation.

† For hypercomplex numbers and quaternions, see for example, *A University Algebra*, D. E. Littlewood. Heinemann's, London (1950).

55. Using quaternions construct the matrix

$$\begin{pmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \\ A_4 & B_4 & C_4 & D_4 \end{pmatrix}$$

with integer elements which has the same sum of squares of elements along the rows, columns and principal diagonals and satisfies the relations

$$A_k A_l + B_k B_l + C_k C_l + D_k D_l = 0 \quad (k \neq l; \quad k, l = 1, 2, 3, 4).$$

§ 2. Elementary transcendental functions

By definition

$$\exp z = e^z = e^x (\cos y + i \sin y);$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}; \qquad \cosh z = \frac{e^z + e^{-z}}{2};$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}; \qquad \sinh z = \frac{e^z - e^{-z}}{2};$$

$$\tan z = \frac{\sin z}{\cos z}; \qquad \tanh z = \frac{\sinh z}{\cosh z};$$

$$\cot z = \frac{\cos z}{\sin z}; \qquad \coth z = \frac{\cosh z}{\sinh z}.$$

56. Using the definition of e^z, prove that:

(1) $e^{z_1} . e^{z_2} = e^{z_1 + z_2}$; (2) $e^{z + 2\pi i} = e^z$.

(3) If $e^{z+\omega} = e^z$ for every z, then

$$\omega = 2\pi k i \qquad (k = 0, \pm 1, \pm 2, \ldots).$$

The relation $\exp i\phi = e^{i\phi} = \cos \phi + i \sin \phi$ (Euler's formula) makes it possible to use the exponential form of notation $z = r e^{i\phi}$ for complex numbers instead of the trigonometric form $z = r (\cos \phi + i \sin \phi)$. In what follows by ϕ is usually understood the principal value of the argument, that is, $-\pi < \phi \leqslant \pi$.

57. Represent the following numbers in the exponential form: $1, \ -1, \ i, \ -i, \ 1+i, \ 1-i, \ -1+i, \ -1-i$.

58. Find $e^{\pm \frac{\pi}{2} i}$; $e^{k\pi i} (k = 0, \pm 1, \pm 2, \ldots)$.

59. Find the moduli and principal values of the arguments of the following complex numbers:

$$e^{2+i};\quad e^{2-3i};\quad e^{3+4i};\quad e^{-3-4i};\quad -ae^{i\phi}\ (a>0,\ |\phi|\leqslant\pi);$$
$$e^{-i\phi}\ (|\phi|\leqslant\pi).$$

60. Find the sums:

(1) $1+\cos x+\cos 2x+\ldots+\cos nx$;
(2) $\sin x+\sin 2x+\ldots+\sin nx$;
(3) $\cos x+\cos 3x+\ldots+\cos(2n-1)x$;
(4) $\sin x+\sin 3x+\ldots+\sin(2n-1)x$;
(5) $\sin x-\sin 2x+\ldots+(-1)^{n-1}\sin nx$.

61. Find the sums:

(1) $\cos\alpha+\cos(\alpha+\beta)+\ldots+\cos(\alpha+n\beta)$;
(2) $\sin\alpha+\sin(\alpha+\beta)+\ldots+\sin(\alpha+n\beta)$.

62. Starting from the definitions of the respective functions, prove:

(1) $\sin^2 z+\cos^2 z=1$; (2) $\sin z=\cos\left(\frac{\pi}{2}-z\right)$;
(3) $\sin(z_1+z_2)=\sin z_1\cos z_2+\cos z_1\sin z_2$;
(4) $\cos(z_1+z_2)=\cos z_1\cos z_2-\sin z_1\sin z_2$;
(5) $\tan 2z=\frac{2\tan z}{1-\tan^2 z}$;
(6) $\cosh(z_1+z_2)=\cosh z_1\cosh z_2+\sinh z_1\sinh z_2$.

63. Prove that if $\cos(z+\omega)=\cos z$ for every z, then $\omega=2\pi k$ $(k=0,\pm1,\pm2,\ldots)$.

64. Prove that:

(1) $\sin iz=i\sinh z$; (3) $\tan iz=i\tanh z$;
(2) $\cos iz=\cosh z$; (4) $\cot iz=-i\coth z$.

65. Express in terms of trigonometric and hyperbolic functions with real arguments the real and imaginary parts and also the moduli of the following functions:

(1) $\sin z$; (4) $\sinh z$;
(2) $\cos z$; (5) $\cosh z$;
(3) $\tan z$; (6) $\tanh z$.

66. Find the real and imaginary parts of the following expressions:

(1) $\cos(2+i)$; (4) $\cot\left(\frac{\pi}{4}-i\log 2\right)$;
(2) $\sin 2i$; (5) $\coth(2+i)$;
(3) $\tan(2-i)$; (6) $\tanh\left(\log 3+\frac{\pi i}{4}\right)$.

67. For each of the functions e^z, $\cos z$, $\sin z$, $\tan z$, $\cosh z$, $\coth z$, find the set of points z, where it assumes:

(1) Real values;

(2) Purely imaginary values.

By definition,

$$\operatorname{Log} z = \log r + i\phi + 2\pi ik \quad (k = 0, \pm 1, \pm 2, \ldots),$$

$$\log z = \log r + i\phi \; (-\pi < \phi \leqslant \pi)$$

($\log z$ is called the *principal value* of the quantity $\operatorname{Log} z$).

68. Evaluate:

(1) $\operatorname{Log} 4$, $\operatorname{Log}(-1)$, $\log(-1)$;

(2) $\operatorname{Log} i$, $\log i$;

(3) $\operatorname{Log} \dfrac{1 \pm i}{\sqrt{2}}$;

(4) $\operatorname{Log}(2-3i)$, $\operatorname{Log}(-2+3i)$.

69*. Find the error in the reasoning leading to I. Bernoulli's paradox: $(-z)^2 = z^2$, hence $2\operatorname{Log}(-z) = 2\operatorname{Log} z$, consequently, $\operatorname{Log}(-z) = \operatorname{Log} z$ (!).

70. The initial value of $\operatorname{Im} f(z)$ at $z = 2$ is assumed to be zero. The point z makes a complete rotation anticlockwise along a circle with centre at the point $z = 0$ and returns to the point $z = 2$. Assuming that $f(z)$ varies continuously during the motion of the point z, find the value of $\operatorname{Im} f(z)$ after the given rotation, if:

(1) $f(z) = 2\operatorname{Log} z$;

(2) $f(z) = \operatorname{Log} 1/z$;

(3) $f(z) = \operatorname{Log} z - \operatorname{Log}(z+1)$;

(4) $f(z) = \operatorname{Log} z + \operatorname{Log}(z+1)$.

By definition, for any complex numbers $a \neq 0$ and α

$$a^\alpha = \exp\{\alpha \operatorname{Log} a\} \tag{1}$$

or, if as usual† by e^z we understand $\exp z$, then

$$a^\alpha = e^{\alpha \operatorname{Log} a}$$

71. Find all the values of the following powers:

(1) $1^{\sqrt{2}}$;

(2) $(-2)^{\sqrt{2}}$;

(3) 2^i;

(4) 1^{-i};

(5) i^i;

(6) $\left(\dfrac{1-i}{\sqrt{2}}\right)^{1+i}$

(7) $(3-4i)^{1+i}$

(8) $(-3+4i)^{1+i}$.

† By eqn.(1) $e^z = \exp\{z \operatorname{Log} e\} = \exp\{z(1+2\pi ik)\}$. However, if nothing is said to the contrary, we shall consider that $k = 0$, that is, as usual $e^z = \exp z$.

72. Show that in the case of a rational index ($\alpha = m/n$) the general definition of the power z^α is identical with the usual definition:

$$z^{\frac{m}{n}} = \left(\sqrt[n]{z}\right)^m$$

(see also problem **5**).

73. Are the sets of values of the following identical:

$$a^{2\alpha},\ (a^\alpha)^2,\ (a^2)^\alpha\ ?$$

By definition, the equation $w = \cos^{-1} z$ is equivalent to the equation $z = \cos w$. The functions $\sin^{-1} z$, $\tan^{-1} z$, $\cot^{-1} z$ and the inverse hyperbolic functions $\cosh^{-1} z$, $\sinh^{-1} z$, $\tanh^{-1} z$, $\coth^{-1} z$ are defined similarly.

74. Prove the following equalities (all the values of the roots are taken):

(1) $\cos^{-1} z = -i \operatorname{Log}\left(z+\sqrt{(z^2-1)}\right)$;

(2) $\sin^{-1} z = -i \operatorname{Log} i\left(z+\sqrt{(z^2-1)}\right)$;

(3) $\tan^{-1} z = \dfrac{i}{2} \operatorname{Log}\left((i+z)/(i-z)\right) = \dfrac{1}{2i} \operatorname{Log} \dfrac{1+iz}{1-iz}$;

(4) $\cot^{-1} z = \dfrac{i}{2} \operatorname{Log}\left((z-i)/(z+i)\right)$;

(5) $\cosh^{-1} z = \operatorname{Log}\left(z+\sqrt{(z^2-1)}\right)$;

(6) $\sinh^{-1} z = \operatorname{Log}\left(z+\sqrt{(z^2+1)}\right)$;

(7) $\tanh^{-1} z = \dfrac{1}{2} \operatorname{Log} \dfrac{1+z}{1-z}$;

(8) $\coth^{-1} z = \dfrac{1}{2} \operatorname{Log} \dfrac{z+1}{z-1}$.

75. Prove that for any value of $\cos^{-1} z$ it is possible to choose a value of $\sin^{-1} z$, such that the sum of these values is equal to $\pi/2$. Prove a similar assertion for $\tan^{-1} z$ and $\cot^{-1} z$.

REMARK. The equalities $\sin^{-1} z + \cos^{-1} z = \pi/2$ and $\tan^{-1} z + \cot^{-1} z = \pi/2$ are always understood in the sense indicated in the present problem.

76. Show that all the values of $\cos^{-1} z$ are contained in the formula

$$\cos^{-1} z = \pm i \operatorname{Log}\left(z+\sqrt{(z^2-1)}\right),$$

where by $\sqrt{(z^2-1)}$ is understood any one of its values.

77. (1) For what values of z are all the values of the functions $\cos^{-1} z$, $\sin^{-1} z$, and $\tan^{-1} z$ real?

(2) For what values of z does the function $\sinh^{-1} z$ assume purely imaginary values?

78. Find all the values of the following functions:

(1) $\sin^{-1} 1/2$; (5) $\tan^{-1}(1+2i)$;
(2) $\cos^{-1} 1/2$; (6) $\cosh^{-1} 2i$;
(3) $\cos^{-1} 2$; (7) $\tanh^{-1}(1-i)$.
(4) $\sin^{-1} i$;

§ 3. Functions of a complex variable

In problems **79–85** it is required to determine the curves defined by the given equations.

79. $z = 1-it$; $0 \leqslant t \leqslant 2$.

80. $z = t+it^2$; $-\infty < t < \infty$.

81. $z = t^2+it^4$; $-\infty < t < \infty$.

82. $z = a(\cos t+i\sin t)$; $\frac{\pi}{2} \leqslant t \leqslant \frac{3\pi}{2}$; $a > 0$.

83. $z = t+(i/t)$; $-\infty < t < 0$.

84. (1) $z = t+i\sqrt{(1-t^2)}$; $-1 \leqslant t \leqslant 1$;

(2) $z = -t+i\sqrt{(1-t^2)}$; $-1 \leqslant t \leqslant 0$ (the arithmetic value of the root is taken).

85. (1) $z = a(t+i-ie^{-it})$; $-\infty < t < \infty$, $a > 0$;

(2) $z = ia+at-ibe^{-t}$; $0 \leqslant t \leqslant 2\pi$, $a > 0$, $b > 0$.

---

86. For the mapping $w = z^2$ it is required:

(1) To find the images of the curves $x = C$, $y = C$, $x = y$, $|z| = R$, $\arg z = \alpha$ and explain which of them are transformed one-one;

(2) To find the originals (in the z-plane) of the curves $u = C$, $v = C$ $(w = u+iv)$.

87. For the mapping $w = \frac{1}{z}$ find:

(1) The images of the curves $x = C$, $y = C$, $|z| = R$, $\arg z = \alpha$, $|z-1| = 1$;

(2) The originals of the curves $u = C$, $v = C$.

88. For the mappings $w = z+\frac{1}{z}$ and $w = z-\frac{1}{z}$ find the images of the circles $|z| = R$.

89. For the transformation $w = z+(1/z)$ find on the z-plane the original of the rectangular net ($u = C$, $v = C$) of the w-plane.

90. What is the circle $|z| = 1$ transformed into in the mapping $w = z/(1-z^2)$?

91. For the mapping $w = e^z$ find:

(1) The images of the curves $x = C$, $y = C$, $x = y$;

(2) The originals of the curves $\varrho = \theta$ $(0 \leqslant \theta < \infty)$.

92. Find the transformations of the rectangular net $(x = C, y = C)$ of the z-plane by means of the functions:

(1) $w = z^2+z$; (2) $w = \coth z$; (3) $w = e^{z^2}$.

93. Into what are the segments of the straight lines $x = C$ and the straight lines $y = C$, which lie in the strip $0 \leqslant y \leqslant \pi$ transformed by means of the function $w = e^z+z$?

94. What corresponds in the z-plane to the polar net $|w| = R$, $\arg w = \alpha$ in the transformations:

(1) $w = e^{\frac{1}{z}}$; (2) $w = e^{z^2}$?

---

95. Find the limit points of the sets:

(1) $z = 1+(-1)^n n/(n+1)$ $(n = 1, 2, \ldots)$;

(2) $z = 1/m+i/n$ (m, n are arbitrary integers);

(3) $z = p/m+iq/n$ (m, n, p, q are arbitrary integers);

(4) $|z| < 1$.

96. Prove that from an infinite bounded sequence of points $\{z_n\}$ it is possible to select a convergent subsequence.

97. Prove the following propositions:

(1) The convergence of the sequence $\{z_n = x_n + iy_n\}$ is equivalent to the simultaneous convergence of the sequences $\{x_n\}$ and $\{y_n\}$.

(2) For the limit $\lim\limits_{n\to\infty} z_n \neq 0$, to exist, it is necessary and sufficient that the limits $\lim\limits_{n\to\infty} |z_n| \neq 0$ and (for a suitable definition of $\arg z_n$) $\lim\limits_{n\to\infty} \arg z_n$ should exist. If $\lim\limits_{n\to\infty} z_n$ is not a negative number it is possible, for example, to take $-\pi < \arg z_n \leqslant \pi$.

In which cases is the convergence of the sequence $\{z_n\}$ equivalent only to the convergence of the sequence $\{|z_n|\}$?

98. On the basis of the statements of problem **97** prove:

(1) $\lim\limits_{n\to\infty} (1+z/n)^n = e^x(\cos y+i\sin y)$;

(2) $\lim\limits_{n\to\infty} [n(\sqrt[n]{(z)}-1)] = \log r+i\phi+2\pi ik$ $(k = 0, 1, 2, \ldots)$.

99*. The function $f(z)$ defined in the neighbourhood of the point z_0 is said to be Heine continuous at the point z_0, if for any sequence $\{z_n\}$, which converges to z_0, the condition $\lim\limits_{n\to\infty} f(z_n) = f(z_0)$ is satisfied; the same function is said to be Cauchy continuous, if for any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that from the inequality $|z-z_0| < \delta$ it follows that $|f(z)-f(z_0)| < \varepsilon$. Prove that these definitions are equivalent (see, for example, [1, Chapter I, § 3.6]).

100. The functions $\dfrac{\operatorname{Re} z}{z}$, $\dfrac{z}{|z|}$, $\dfrac{\operatorname{Re}(z^2)}{|z|^2}$, $\dfrac{z \operatorname{Re} z}{|z|}$ are defined for $z \neq 0$. Which of them can be defined at the point $z = 0$ in such a way that they are continuous at this point?

101. Are the functions

$$(1)\ \frac{1}{1-z}, \qquad (2)\ \frac{1}{1+z^2}$$

continuous inside the unit disk ($|z| < 1$)? Are they uniformly continuous?

102. (1) Prove that the function $e^{-\frac{1}{|z|}}$ is uniformly continuous in the disk $|z| \leqslant R$ with the point $z = 0$ deleted.

(2) Is the function $e^{\frac{1}{z^2}}$ uniformly eontinuous in the same domain?

(3) Is the function $e^{\frac{1}{z^2}}$ uniformly continuous in the sector $0 < |z| \leqslant R$, $|\arg z| \leqslant \pi/6$?

103. The function $w = e^{-\frac{1}{z}}$ is defined everywhere except at the point $z = 0$. Prove that:

(1) in the semicircle $0 < |z| \leqslant 1$, $|\arg z| \leqslant \pi/2$ this function is bounded but not continuous;

(2) inside the same semicircle the function is continuous but not uniformly eontinuous;

(3) in the sector $0 < |z| \leqslant 1$, $|\arg z| \leqslant \alpha < \pi/2$ the function is uniformly continuous.

104. The funetion $f(z)$ is uniformly continuous in the disk $|z| < 1$. Prove that for any point ζ on the circle $|z| = 1$ and any sequence $z_n \to \zeta$, $|z_n| < 1$ there exists the limit $\lim\limits_{n\to\infty} f(z_n)$. Prove also that this limit does not depend on the choice of the sequence and that if $f(z)$ is defined on the boundary circle by means of passing to the limit, it is continuous throughout the whole of the closed disk $|z| \leqslant 1$.

§ 4. Analytic and harmonic functions

The Cauchy–Riemann equations

105. Verify that the Cauchy–Riemann equations are satisfied for the functions z^n, e^z, $\cos z$, $\operatorname{Log} z$ and prove that

$$(z^n)' = nz^{n-1}, \quad (e^z)' = e^z,$$

$$(\cos z)' = -\sin z, \quad (\log z)' = \frac{1}{z}.$$

106. Write down the Cauchy–Riemann equations in polar coordinates.

107. Prove that if $f(z) = u+iv$ is an analytic function and **s** and **n** are perpendicular vectors, the rotation from the vector **s** to the vector **n** through a right angle being anticlockwise, then

$$\frac{\partial u}{\partial s} = \frac{\partial v}{\partial n} \quad \text{and} \quad \frac{\partial u}{\partial n} = -\frac{\partial v}{\partial s}$$

($\partial/\partial s$ and $\partial/\partial n$ are the derivatives of the function of two real variables with respect to the corresponding directions).

108. Prove that the function $f(z) = \bar{z}$ is not differentiable anywhere.

109. Prove that the function $w = z \operatorname{Re} z$ is differentiable only at the point $z = 0$; find $w'(0)$.

110. Prove that the Cauchy–Riemann equations are satisfied for the function $f(z) = \sqrt{(|xy|)}$ at the point $z = 0$ but the derivative does not exist.

111. Prove the following assertions:

(1) If the limit

$$\lim_{\Delta z \to 0} \left[\operatorname{Re} \frac{\Delta w}{\Delta z} \right]$$

exists for the function $w = f(z)$ then the partial derivatives u_x and v_y exist and are equal.

(2) If the limit $\lim\limits_{\Delta z \to 0} \left[\operatorname{Im} \dfrac{\Delta w}{\Delta z} \right]$ exists then the partial derivatives u_y and v_x exist and $u_y = -v_x$.

(3) If it is assumed in advance that the functions u and v are differentiable then the existence of either of the limits mentioned in parts (1) and (2) ensures the existence of the other and consequently the differentiability of the function $f(z)$.

112. At the point z_0 the function $w = f(z)$ possesses the following properties: (1) u, v are differentiable; (2) the limit $\lim\limits_{\Delta z \to 0} \left| \frac{\Delta w}{\Delta z} \right|$ exists. Prove that either $f(z)$ or $\overline{f(z)}$ is differentiable at the point z_0.

113. At the point z_0 the function $w = f(z)$ possesses the following properties: (1) the functions u, v are differentiable; (2) the limit $\lim\limits_{\Delta z \to 0} \arg \frac{\Delta w}{\Delta z}$ exists. Prove that $f(z)$ is differentiable at the point z_0.

114. $f(z) = u + iv = \varrho e^{i\theta}$ is an analytic function. Prove that if one of the functions u, v, ϱ, θ is identically equal to a constant, then the function $f(z)$ is also a constant.

Harmonic functions

A function $u(x, y)$ which in some domain possesses continuous partial derivatives up to the second order inclusive and satisfies Laplace's equation

$$\Delta u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

is said to be a *harmonic function.* Two harmonic functions $u(x, y)$ and $v(x, y)$ connected by the Cauchy–Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

are said to be *conjugate.*

115. Prove the following propositions:

(1) A linear combination of harmonic functions $\sum\limits_{i=1}^{n} c_i u_i(x, y)$ is a harmonic function.

(2) If the inversion transformation

$$x = \frac{\xi}{\xi^2 + \eta^2}, \qquad y = \frac{\eta}{\xi^2 + \eta^2},$$

is applied to the argument of the harmonic function $u(x, y)$ the transformed function is harmonic.

(3) If the transformation

$$x = \phi(\xi, \eta), \qquad y = \psi(\xi, \eta),$$

where ϕ and ψ are conjugate harmonic functions is applied to the argument of the harmonic function $u(x, y)$, the transformed function is harmonic. (Hence, in particular, the preceding proposition follows.)

(4) Let $u(x, y)$ and $v(x, y)$ be conjugate harmonic functions and let the Jacobian $\dfrac{\partial(u, v)}{\partial(x, y)}$ be different from zero in a certain domain. Then the inverse functions $x(u, v)$ and $y(u, v)$ are also harmonic and conjugate.

116. (1) Prove that every function $u(x, y)$ harmonic in the simply connected domain G has a family of conjugate harmonic functions, differing from one another by an additive constant

$$v(x, y) = \int_{(x_0, y_0)}^{(x, y)} -\frac{\partial u}{\partial y}\mathrm{d}x + \frac{\partial u}{\partial x}\mathrm{d}y + C.$$

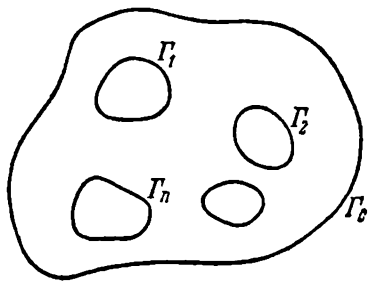

FIG. 1

(2) Prove that if the domain G is multiply connected and bounded by an external contour Γ_0 and by internal contours Γ_1, $\Gamma_2, \ldots, \Gamma_n$ (Fig. 1) (each of which can be shrunk to a point), then the function $v(x, y)$ may turn out to be many-valued and the general formula for its values will be of the form

$$v(x, y) = \int_{(x_0, y_0)}^{(x, y)} -\frac{\partial u}{\partial y}\mathrm{d}x + \frac{\partial u}{\partial x}\mathrm{d}y + \sum_{k=1}^{n} m_k \pi_k + C.$$

The integral is taken along a path which lies in the domain G, the m_k are integers and

$$\pi_k = \int_{\gamma_k} -\frac{\partial u}{\partial y}\mathrm{d}x + \frac{\partial u}{\partial x}\mathrm{d}y,$$

where the γ_k are simple closed contours each of which contains within it a single connected part of the boundary (Γ_k) (the numbers π_k are known as the *periods of the integral* or the *cyclical constants*).

For the function $v(x, y)$ to be single-valued it is necessary and sufficient that all the numbers π_k should be zero.

REMARK. The contour Γ_0 may not exist provided that the function $u(x, y)$ is harmonic at the point at infinity. By definition this means that the function $U(\xi, \eta)$, obtained from the function $u(x, y)$ by the inversion transformation (see problem **115**, (2)), will be harmonic at the coordinate origin. It can be proved that in this case $\sum_{k=1}^{n} \pi_k = 0$.

117. Assuming it to be known that an analytic function is infinitely differentiable, prove the following theorems:

(1) The real and imaginary parts of the analytic function $f(z) = u+iv$ are conjugate harmonic functions.

(2) The derivatives (of any order) of a harmonic function are also harmonic functions.

118. (1) Is u^2 a harmonic function if u is a harmonic function?

(2) Let u be a harmonic function. For what functions f is the function $f(u)$ also harmonic?

119. Are $|f(z)|$, $\arg f(z)$, $\log|f(z)|$, harmonic functions if $f(z)$ is an analytic function?

120. Transform Laplace's operator $\Delta u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$ into the polar system of coordinates (r, ϕ) and find the solution of Laplace's equation $\Delta u = 0$, which depends only on r.

121. Calculate for $n = 1, 2, 3, 4$ the harmonic polynomials $p_n(x, y)$ and $q_n(x, y)$, defined by the equation $z^n = p_n+iq_n$. Write down the general form of p_n and q_n in polar coordinates.

Using the formulae of problem **116** in problems **122–126** find the functions conjugate to the given harmonic functions in the domains indicated.

122. $u(x, y) = x^2-y^2+x, \quad 0 \leqslant |z| < \infty.$

123. $u(x, y) = \dfrac{x}{x^2+y^2}, \quad 0 < |z| \leqslant \infty.$

124. $u(x, y) = \dfrac{1}{2}\log(x^2+y^2)$;

(a) in the domain obtained from the plane by the deletion of the semi-axis: $y = 0, -\infty < x \leqslant 0$;

(b) in the plane with the coordinate origin deleted $(0<|z|<\infty)$.

125. $u(x, y) = \dfrac{1}{2}\log(x^2+y^2) - \log[(x-1)^2+y^2]$:

(a) in the plane with the points $z = 0$ and $z = 1$ deleted;

(b) in the plane with the segment $y = 0$, $0 \leqslant x \leqslant 1$ of the real axis deleted;

(c) in the plane with the ray $y = 0$, $1 \leqslant x < \infty$, deleted.

126. $u(x, y) = \frac{1}{2} \sum_{k=1}^{n} \alpha_k \log[(x-x_k)^2+(y-y_k)^2]$;

(a) in the plane with the deletion of the points $z_1, z_2, \ldots, z_n$ ($z_k = x_k+iy_k$; $z_l \neq z_j$);

(b) in the plane with the deletion of a simple curve (that is, one free from self-intersections) formed of rectilinear segments, connecting the given points.

127. Do analytic functions $f(z) = u+iv$ exist for which:

(1) $u = \frac{x^2-y^2}{(x^2+y^2)^2}$;

(2) $v = \log(x^2+y^2)-x^2+y^2$;

(3) $u = e^{\frac{y}{x}}$?

In problems **128–131** find the analytic functions $f(z) = u+iv$ from the given real or imaginary part.

128. $u = x^2-y^2+5x+y-\frac{y}{x^2+y^2}$.

129. $u = e^x(x\cos y-y\sin y)+2\sin x\sinh y+x^3-3xy^2+y$.

130. $v = 3+x^2-y^2-\frac{y}{2(x^2+y^2)}$.

131. $v = \log(x^2+y^2)+x-2y$.

REMARK. See also problems **569–572**.

In problems **132–139** explain whether harmonic functions of the given form (different from a constant) exist and if they exist find them.

132. $u = \phi(x)$.

133. $u = \phi(ax+by)$ (a and b are real numbers).

134. $u = \phi\left(\frac{y}{x}\right)$.

135. $u = \phi(xy)$.

136. $u = \phi(x^2+y^2)$.

137. $u = \phi\left(\frac{x^2+y^2}{x}\right)$.

138. $u = \phi[x+\sqrt{(x^2+y^2)}]$.

139. $u = \phi(x^2+y^2)$.

In problems **140–143** prove the existence of and find the analytic functions $f(z)$ from the given modulus or argument.

140. $\varrho = (x^2+y^2)e^x$.

141. $\varrho = e^{r^2\cos 2\phi}$.

142. $\theta = xy$.

143. $\theta = \phi + r\sin\phi$.

144. Prove that for the family of curves $\phi(x, y) = C$, where ϕ is a twice continuously differentiable function, to be the family of level lines of some harmonic function it is necessary and sufficient that the ratio $\Delta\phi/(\text{grad}\,\phi)^2$ should depend only on ϕ.

HINT. As a preliminary establish that the required harmonic function is of the form $u = f[\phi(x, y)]$.

In problems **145–149** find analytic functions of which the real part, the imaginary part, the modulus or the argument preserves a constant value along any line of the corresponding family.

145. $x = C$.

146. $y = C$.

147. $y = Cx$.

148. $x^2+y^2 = C$.

149. $x^2+y^2 = Cx$.

The geometrical meaning of the modulus and argument of a derivative

150. Mappings are made using the functions $w = z^2$ and $w = z^3$. For each function find the angle of rotation (θ) of the directions issuing from the point z_0, and the magnification (k) at the following points:

(1) $z_0 = 1$; (2) $z_0 = -1/4$; (3) $z_0 = 1+i$; (4) $z_0 = -3+4i$.

151. Which part of the plane is compressed and which is stretched if the mapping is effected by the functions:

(1) $w = z^2$;
(2) $w = z^2+2z$;
(3) $w = 1/z$;
(4) $w = e^z$;
(5) $w = \log(z-1)$?

152. The domain G is mapped by means of the function $f(z)$ conformally and one-one onto the domain G'. Find formulae for the calculation of the area S of the domain G' and the length L of the arc onto which some arc l belonging to the domain G is mapped.

153. Find the length L of the spiral onto which the function e^z maps the segment $y = x$, $0 \leqslant x \leqslant 2\pi$ (see problem **91**).

154. Find the area of the domain onto which the function e^z maps the rectangle $1 \leqslant x \leqslant 2$, $0 \leqslant y \leqslant 4$.

155. Find the domain D, onto which the function e^z maps the rectangle $1 \leqslant x \leqslant 2$, $0 \leqslant y \leqslant 8$. Calculate the area of the domain D by means of the formula obtained in the solution of problem **152** and explain why this formula gives an incorrect result.

CHAPTER II

CONFORMAL MAPPINGS CONNECTED WITH ELEMENTARY FUNCTIONS

§ 1. Linear functions

Linear functions

156. Find the linear function which maps the triangle with vertices at the points 0, 1, i onto a similar triangle with vertices at 0, 2, $1+i$.

157. Find the linear transformation with fixed point $1+2i$ which transforms the point i into the point $-i$.

158. For the transformations given find the finite fixed point z_0 (if it exists), the angle of rotation round it, θ, and the magnification k. Reduce these transformations to the canonical form $w-z_0 = \lambda(z-z_0)$.

(1) $w = 2z+1-3i$;
(2) $w = iz+4$;
(3) $w = z+1-2i$;
(4) $w-w_1 = a(z-z_1)$ $(a \neq 0)$;
(5) $w = az+b$ $(a \neq 0)$.

159. Find the general form of the linear transformation which transforms:

(1) The upper half-plane into itself;
(2) The upper half-plane into the lower half-plane;
(3) The upper half-plane into the right half-plane;
(4) The right half-plane into itself.

Show that in each case the transformation is uniquely determined by specifying a single pair of corresponding interior points or two pairs of boundary points.

160. Find the general form of the linear transformation which transforms:

(1) The strip $0 < x < 1$ into itself;
(2) The strip $-2 < y < 1$ into itself;
(3) The strip bounded by the straight lines $y = x$ and $y = x-1$ into itself.

Explain which pairs of points can correspond to one another in these mappings and in which case this correspondence determines the mapping uniquely.

161. Find the linear function $w(z)$, which maps the strip, contained between given straight lines onto the strip $0 < u < 1$ with the given normalisation:

(1) $x = a, \quad x = a+h; \quad w(a) = 0;$

(2) $x = a, \quad x = a+h; \quad w\left(a+\frac{h}{2}\right) = \frac{1}{2}+i;$

$$\operatorname{Im} w\left(a+\frac{h}{2}+i\right) < 1.$$

(3) $y = kx, \quad y = kx+b; \quad w(0) = 0;$

(4) $y = kx+b_1, \quad y = kx+b_2; \quad w(b_1) = 0.$

162. Find the linear function which maps the circle $|z| < 1$ onto the circle $|w-w_0| < R$ and is such that the centres of the circles correspond to one another and the horizontal diameter passes into a diameter making an angle α with the direction of the real axis.

Bilinear functions

163. For the function $w = 1/z$ find the images of the following curves:

(1) The family of circles $x^2+y^2 = ax$;

(2) The family of circles $x^2+y^2 = by$;

(3) The pencil of parallel straight lines $y = x+b$;

(4) The pencil of straight lines $y = kx$;

(5) The pencil of straight lines passing through a given point $z_0 \neq 0$;

(6) The parabola $y = x^2$.

164. Explain into what the function $w = \frac{1}{z-z_0}+h$ transforms:

(1) The rectangular net $x = C$, $y = C$;

(2) The polar net $|z-z_0| = R$, $\arg(z-z_0) = \alpha$.

165. Given the function $w = \frac{z-z_1}{z-z_2}$:

(1) Prove that the original of the family $|w| = \lambda$ $(0 < \lambda < \infty)$ is a family of circles (the circles of Apollonius). For a given λ find the radius and the position of the centre of the corresponding circle in the z-plane.

(2) Find the original of the ray $\arg w = \theta$.

(3) Construct the net in the z-plane which corresponds to the polar net in the w-plane.

(4) Find the domain of the z-plane which corresponds to the semicircle $|w| < 1$, $\operatorname{Im} w > 0$.

In problems **166–170** explain what the domains indicated are transformed into with the given mapping functions.

166. The quadrant $x > 0$, $y > 0$; $\quad w = \dfrac{z-i}{z+i}$.

167. The semicircle $|z| < 1$, $\operatorname{Im} z > 0$; $w = \dfrac{2z-i}{2+iz}$.

168. The angle $0 < \phi < \pi/4$; $w = z/(z-1)$.

169. The strip $0 < x < 1$; (1) $w = (z-1)/z$; (2) $w = (z-1)/(z-2)$.

170. The ring $1 < |z| < 2$; $w = z/(z-1)$.

171. Map onto the vertical strip $0 < \operatorname{Re} w < 1$:

(1) The half-plane $\operatorname{Re} z > 0$ with the disk $|z-d/2| \leqslant d/2$ deleted;

(2) The lune contained between the circles

$$|z-d_1/2| = d_1/2, \quad |z-d_2/2| = d_2/2 \qquad (d_1 < d_2);$$

(3) The exterior of the circles $|z+d_1/2| = d_1/2$, $|z-d_2/2| = d_2/2$ so that $w(d_2) = 0$.

172. Find the bilinear function which transforms the points -1, i, $1+i$ respectively into the points:

(1) 0, $2i$, $1-i$; $\qquad$ (2) i, ∞, 1.

173. Find the bilinear function which transforms the points -1, ∞, i respectively into the points:

(1) i, 1, $1+i$; $\quad$ (2) ∞, i, 1; $\quad$ (3) 0, ∞, 1.

174. Find bilinear functions from the following conditions:

(1) The points 1 and i are fixed and the point 0 passes into the point -1;

(2) The points $1/2$ and 2 are fixed and the point $5/4+3i/4$ passes into ∞;

(3) The point i is a double fixed point and the point 1 passes into ∞.

175. Find the bilinear function which transforms the points -1, 0, 1 respectively into the points 1, i, -1, and explain what the upper half-plane becomes in this mapping.

176. Find the general form of the bilinear transformation which transforms:

(1) The upper half-plane into itself;

(2) The upper half-plane into the lower half-plane;

(3) The upper half-plane into the right half-plane.

177. Find the mapping of the upper half-plane into itself with the given normalisation:

(1) $w(0) = 1$, $w(1) = 2$, $w(2) = \infty$;

(2) $w(0) = 1$, $w(i) = 2i$.

REMARK. For mappings of the upper half-plane into itself for other normalisations see problem 188.

178. Find the function $w(z)$ which maps the circle $|z| < R$ onto the right half-plane $\operatorname{Re} w > 0$ in such a way that $w(R) = 0$, $w(-R) = \infty$, $w(0) = 1$. What is the image of the upper semicircle in this mapping?

Two points P_1, P_2 are said to be symmetrical with respect to the circle K with centre O and radius R, if they lie on the same ray issuing from O and

$$OP_1 . OP_2 = R^2.$$

179. Find the points symmetrical to the point $2+i$ with respect to the circles:

(1) $|z| = 1$; (2) $|z-i| = 3$.

180. Find the symmetrical image with respect to the unit circle of the following curves:

(1) $|z| = 1/2$;

(2) $|z-1| = 1$;

(3) $y = 2$;

(4) $|z-z_0| = |z_0|$ $(z_0 = x_0+iy_0)$;

(5) $|z-z_0| = \sqrt{(|z_0|^2-1)}$ $(|z_0| > 1)$;

(6) the hyperbola $x^2-y^2 = 1$;

(7) The perimeter of the rectilinear triangle with vertices z_1, z_2, $z_3 (z_i \neq 0)$.

181. Prove that for the symmetry of the points P_1 and P_2 with respect to K it is necessary and sufficient that one of the following two conditions should be satisfied:

(1) Every circle K_1 passing through the points P_1, P_2, is orthogonal to K;

(2) For all points M of the circle K

$$\frac{MP_1}{MP_2} = \text{const}$$

(that is, K is a circle of Apollonius with respect to the points P_1 and P_2).

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182. The function

$$w = e^{i\alpha} \frac{z-\beta}{z-\bar{\beta}} \qquad (\beta = a+ib,\ b > 0)$$

maps the upper half-plane onto the unit disk.

(1) Find $\arg w(x) = \theta(x)$;

(2) Find $w'(\beta)$;

(3) Explain what part of the upper half-plane is compressed in this mapping and what part is stretched.

183. Map the upper half-plane $\operatorname{Im} z > 0$ onto the unit disk $|w| < 1$ in such a way that:

(1) $w(i) = 0$, $\arg w'(i) = -\pi/2$;

(2) $w(2i) = 0$, $\arg w'(2i) = 0$;

(3) $w(a+bi) = 0$, $\arg w'(a+bi) = \theta$ $(b > 0)$.

184. Map the upper half-plane $\operatorname{Im} z > 0$ onto the disk $|w-w_0| < R$ so that the point i passes into the centre of the disk and the derivative at this point is positive.

185. Map the disk $|z| < 2$ onto the half-plane $\operatorname{Re} w > 0$ in such a way that $w(0) = 1$, $\arg w'(0) = \pi/2$.

186. Map the disk $|z-4i| < 2$ onto the half-plane $v > u$ so that the centre of the disk passes into the point -4, and the point $2i$ on the circumference becomes the coordinate origin.

187. Find the general form of the bilinear function $w(z)$ which maps the disk $|z| < R$ onto the right half-plane $\operatorname{Re} w > 0$ in such a way that $w(z_1) = 0$, $w(z_2) = \infty$, where z_1, z_2, are specified points on the circumference $|z| = R$. Construct the family of curves in the circle $|z| < R$ corresponding to the polar net in the half-plane $\operatorname{Re} w > 0$.

HINT. It is possible to use either the general form of the transformation of the half-plane into a disk or the general form of the bilinear transformation for three pairs of corresponding points and the result of problem 8.

188. Find the function which maps the upper half-plane onto itself so that

$$w(a) = b, \quad \arg w'(a) = \alpha \quad (\operatorname{Im} a > 0,\ \operatorname{Im} b > 0).$$

HINT. As a preliminary map both specimens of the half-plane onto the unit disk with the corresponding normalisation.

189. Map the upper half-plane onto the lower one so that $w(a) = \bar{a}$ and $\arg w'(a) = -\pi/2$ $(\operatorname{Im} a > 0)$.

190. For the function

$$w = e^{i\alpha}\frac{z-a}{1-\bar{a}z} \quad (|a| < 1)$$

which maps the unit disk onto itself:

(1) Find $\arg w(e^{i\phi}) = \theta(\phi)$;

(2) Find $w'(0)$ and $w'(a)$;

(3) Explain which part of the unit disk is compressed in this mapping and which is stretched;

(4) Find $\max |dw/dz|$ and $\min |dw/dz|$ for $|z| \leqslant 1$.

191. Map the disk $|z|<1$ onto the disk $|w|<1$ so that:

(1) $w(1/2) = 0$, $\arg w'(1/2) = 0$;

(2) $w(i/2) = 0$, $\arg w'(i/2) = \pi/2$;

(3) $w(0) = 0$, $\arg w'(0) = -\pi/2$;

(4) $w(a) = a$, $\arg w'(a) = \alpha$.

192. Map the disk $|z|<R_1$ onto the disk $|w|<R_2$ so that $w(a) = b$, $\arg w'(a) = \alpha$ ($|a|<R_1$, $|b|<R_2$).

193. Map the disk $|z| < 1$ onto the disk $|w-1|<1$ so that $w(0) = 1/2$ and $w(1) = 0$.

194. Map the disk $|z-2|<1$ onto the disk $|w-2i|<2$ so that $w(2) = i$ and $\arg w'(2) = 0$.

195. Find the general form of the bilinear function $w(z)$, which maps the disk $|z|<R$ onto itself subject to the following conditions:

(1) $w(a) = 0$ ($|a|<R$);

(2) $w(a) = b$ ($|a|<R$, $|b|<R$);

(3) $w(\pm R) = \pm R$.

196. Map the disk $|z|<1$ onto itself so that the specified points z_1, z_2, within the disk pass into the points $\pm a$ ($|a|<1$); find $|a|$.

197. Map the disk $|z|<1$ onto itself so that the real axis segment: $y = 0$, $0 \leqslant x \leqslant a$ ($a<1$) passes into a real axis segment symmetrical about the coordinate origin. Find the length of the transformed segment.

198. Prove that in the mapping of a disk onto a disk the linear transformation is uniquely determined by the specification of one interior and one boundary point.

199. The unit disk is mapped onto itself in such a way that the point $z_0 \neq 0$ passes into the centre of the disk. Prove that then a unit semicircle is mapped onto a unit semicircle if and only if its ends lie on a diameter passing through the point z_0.

200. Construct the mapping of the unit disk onto itself in which the original of the centre is on the real axis and the arc $0 \leqslant \phi \leqslant \pi/2$ of the unit circle is mapped onto the following arcs:

(1) $0 \leqslant \theta \leqslant \pi/2$; (2) $0 \leqslant \theta \leqslant \pi/3$; (3) $\pi/2 \leqslant \theta \leqslant 7\pi/6$.

§ 2. Supplementary questions of the theory of linear transformations

Canonical forms of linear transformations

A bilinear transformation with one fixed point is said to be *parabolic*. A

parabolic transformation can be written in the canonical form

$$\frac{1}{w-z_0} = \frac{1}{z-z_0} + h,$$

if $z_0 \neq \infty$, or

$$w = z + h$$

if $z_0 = \infty$.

A bilinear transformation with two distinct fixed points z_1 and z_2 has the canonical form

$$\frac{w-z_1}{w-z_2} = k\frac{z-z_1}{z-z_2},$$

if $z_1 \neq \infty$, $z_2 \neq \infty$, or

$$w - z_1 = k\,(z-z_1)$$

if $z_2 = \infty$; the transformation with two distinct fixed points is *hyperbolic* if $k > 0$, *elliptic* if $k = e^{i\alpha}$ and $\alpha \neq 0$ and *loxodromic* if $k = ae^{i\alpha}$, where $a \neq 1$ and $\alpha \neq 0$ (α and a real numbers, $a > 0$).

201. Prove the following assertions:

(1) The general bilinear transformation

$$w = \frac{az+b}{cz+d}$$

can be reduced to the form

$$w = \frac{\alpha z+\beta}{\gamma z+\delta}, \quad \text{where} \quad \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} = 1.$$

(2) If $\alpha+\delta$ is a real number, the transformation is elliptic when $|\alpha+\delta| < 2$, hyperbolic when $|\alpha+\delta| > 2$, and parabolic when $|\alpha+\delta| = 2$.

(3) If $\operatorname{Im}(\alpha+\delta) \neq 0$, then the transformation is loxodromic.

202. Prove that if a linear transformation has two fixed points the product of the derivatives at these points is equal to unity.

203. Find the circles which in the parabolic transformation

$$\frac{1}{w-z_0} = \frac{1}{z-z_0} + h$$

pass into themselves.

204. Find the general form of the parabolic transformation of the disk $|z| < R$ into itself if the point R is fixed.

205. Prove the following properties of the hyperbolic transformation:

(1) Any circle passing through the two fixed points is transformed into itself, the direction of traversal being preserved.

(2) Any circle orthogonal to the circles passing through the fixed points is transformed into a circle having the same property. (This property follows directly from property (1).)

HINT. As a preliminary consider the case when the fixed points are 0 and ∞.

206. Prove the following properties of the elliptic transformation:

(1) Any circle orthogonal to the circles passing through two fixed points is transformed into itself with preservation of the direction of traversal.

(2) A circular arc connecting the fixed points passes into a circular arc connecting the fixed points and making an angle α with the first arc ($\alpha = \arg k$).

207. (1) Prove that in the loxodromic transformation the properties (2) of the hyperbolic (see problem **205**) and of the elliptic transformation (see problem **206**) are preserved.

(2) Prove that, provided that $\alpha \neq \pi$ ($\alpha = \arg k$), fixed circles do not exist in the loxodromic transformation. Prove that if $\alpha = \pi$, then the circles passing through the fixed points transform into themselves with change of the direction of traversal.

208. Prove that in the loxodromic transformation $w = ae^{i\alpha}z$ the logarithmic spirals $r = Ae^{\frac{\log a}{\alpha}\phi}$ ($A > 0$) pass into themselves.

209. Prove that the linear transformation

$$w = e^{i\lambda}\frac{z-a}{1-\bar{a}z} \qquad (a = |a|e^{i\alpha}, \quad |a| < 1),$$

which transforms the unit circle into itself can only be elliptic, hyperbolic or parabolic. Explain for what values of a each of the given cases holds. Find the fixed points of the transformation and reduce it to the canonical form.

Some approximate formulae for linear transformations

210. The upper half-plane is mapped onto the unit disk so that the point $z = hi$ ($h > 0$) passes into the centre of the circle. Find the length Γ of the image of the segment $[0, a]$ of the real axis ($a > 0$) and obtain linear approximate formulae for Γ for small a/h and for small h/a.

211. The unit disk is mapped onto itself in such a way that the original of the centre of the disk, the point x_0, is on the real axis. Find the length Γ of the image of the arc $0 \leqslant \phi \leqslant \gamma$ of the unit

circle ($\gamma \leqslant \pi$). How does the quantity Γ/γ vary depending on the sign of x_0?

212. With the conditions of problem **211** obtain the formulae:

(1) $\Gamma = \dfrac{1+x_0}{1-x_0}\gamma + O(\gamma^3)$ for small γ;

(2) $\Gamma = \pi - \varepsilon \cot\dfrac{\gamma}{2} - \dfrac{\varepsilon^2}{2}\cot^2\dfrac{\gamma}{2} + O(\varepsilon^3)$ for small ε, where $\varepsilon = 1-x_0$.

213. The unit disk is mapped onto itself so that the point $z_0 = r_0 e^{i\phi}$ passes into the eentre. The points $z_1 = e^{i\phi_1}$ and $z_2 = e^{i\phi_2}$ lie on opposite sides of the diameter passing through z_0 ($\phi_0 < \phi_1 < \phi_2 \leqslant \phi_0 + \pi$). Assuming that the point z_0 is situated close to the unit circle, prove that the length Γ of the image of the arc $\phi_1 \leqslant \phi \leqslant \phi_2$ of the unit circle satisfies the formula

$$\Gamma = \varepsilon\left[\cot\frac{\phi_1-\phi_2}{2} - \cot\frac{\phi_2-\phi_0}{2}\right] + \frac{\varepsilon^2}{2}\left[\cot^2\frac{\phi_1-\phi_0}{2} - \cot^2\frac{\phi_2-\phi_0}{2}\right] + O(\varepsilon^3)$$

where $\varepsilon = 1-r_0$.

Mappings of simply connected domains

214. Prove that if the linear mapping of the disk $|z| < 1$ onto itself does not reduce to a rotation, then no concentric ring with centre at the coordinate origin passes into a concentric ring.

REMARK. This proposition is a particular case of the following theorem: For the conformal mapping of the ring $r_1 < |z| < r_2$ onto the ring $R_1 < |w| < R_2$ it is necessary and sufficient that the condition $R_2/R_1 = r_2/r_1$ should be satisfied. In this case the mapping function can only have two forms: $w = az$ or $w = a/z$. The mapping is uniquely determined by the specification of a single pair of boundary points (see, for example, [3, Chapter II, § 3]).

215. (1) Map the ring $2 < |z| < 5$ onto the ring $4 < |w| < 10$ so that $w(5) = -4$.

(2) Map the ring $1 < |z-2i| < 2$ onto the ring $2 < |w-3+2i| < 4$ so that $w(0) = -1-2i$.

The following theorem holds†:

Every doubly connected domain, the boundaries of which do not shrink

† See, for example, the paper: M. V. KELDYSH, (1939) Conformal representations of multiply connected domains (Konformnyye otobrazheniya mnogosvyaznykh oblastei), *Uspekhi matem. nauk*, vol. VI, 90–119.

to points, can be conformally mapped onto a concentric ring with well-defined ratio μ of the radii of the outer and inner circles (μ is the *modulus* of the doubly connected domain).

216. Map the semicircle $\operatorname{Re} z > 0$ with the disk $|z-h| < R$ $(h > R)$ deleted onto the ring $\varrho < |w| < 1$ so that the imaginary axis passes into the circle $|w| = 1$. Find ϱ.

HINT. Construct the circle with centre at the coordinate origin and orthogonal to the circle $|z-h| = R$; then find the linear transformation, which transforms the real axis and the constructed circle into two intersecting (orthogonally) straight lines and verify that the given domain is then mapped into the concentric ring. Prove that the centre of this ring is coincident with the coordinate origin if the points of intersection of the constructed circle and the real axis pass into 0 and ∞.

217. Map the half-plane $\operatorname{Re} z > 0$ with the disk $|z-h| < 1$, $h > 1$ deleted, onto the ring $1 < |w| < 2$. Find h.

218. Map the eccentric ring bounded by the circles $|z-3| = 9$, $|z-8| = 16$ onto the ring $\varrho < |w| < 1$. Find ρ.

219. Map the doubly connected domain bounded by the circles $|z-z_1| = r_1$, $|z-z_2| = r_2$ ($|z_2-z_1| > r_1+r_2$ or $|z_2-z_1| < |r_2-r_1|$) onto a concentric circular ring with centre at the coordinate origin. Find the modulus (μ) of the domain.

HINT. Find a pair of points symmetrical with respect to both circles and map one of them onto 0 and the other onto ∞.

REMARK. It is easily verified that the methods of solution recommended in the hints to problems **216** and **219** are the same.

220. Using the solution of the preceding problem find the moduli of the doubly connected domains bounded by the given circles:

(1) $|z-1| = 2$, $|z+1| = 5$;

(2) $|z-3i| = 1$, $|z-4| = 2$.

Group properties of bilinear transformations

The transformation $T(z) = T_2[T_1(z)]$ is said to be the *product* of T_1 and T_2 and is written in the form $T = T_2T_1$ (the order is important since, generally speaking, $T_2T_1 \neq T_1T_2$). The set G of transformations T forms a *group* if it contains the product of every two transformations belonging to it and together with the transformation T contains the transformation T^{-1} inverse to it. A group consisting of the powers T^n and T^{-n} of a single transformation T is said to be *cyclical*. If the group G is formed from the transformations $T_1, T_2, \ldots, T_n$ by the construction of all the inverse transformations and of all possible products of the given transformations and their inverses, these transformations are said to be *generators* of the group G. The points obtained from the fixed point z by means of all the transformations of the group G are said to be *equivalent* or *congruent* with respect to the group G.

The *fundamental domain* of the group is an open set (connected or disconnected) which does not contain a single pair of points equivalent to one another

with respect to the given group, and in the neighbourhood of every boundary point of which there are points equivalent to the points of the set.

221. Let T_i be the linear transformations:

$$T_i(z) = \frac{a_i z + b_i}{c_i z + d_i}, \quad \Delta_i = \begin{vmatrix} a_i & b_i \\ c_i & d_i \end{vmatrix} \neq 0 \quad (i = 1, 2, \ldots).$$

Prove the following assertions:

(1) $T = T_1 T_2$ is a linear transformation with determinant $\Delta = \Delta_1 \Delta_2$.

(2) The product of the transformations is associative, that is,

$$(T_3 T_2) T_1 = T_3 (T_2 T_1).$$

(3) Every transformation T_i has an inverse T_i^{-1}, that is,

$$T_i T_i^{-1} = T_i^{-1} T_i = J,$$

where $J(z) \equiv z$ is the identical transformation.

(4) The product of transformations, generally speaking, is not commutative (give examples).

222. Prove that the transformations

$$T_1 = z, \quad T_2 = \frac{1}{z} \quad T_3 = 1 - z, \quad T_4 = \frac{1}{1-z},$$

$$T_5 = \frac{z-1}{z}, \quad T_6 = \frac{z}{z-1}$$

form a group (the group of *anharmonic ratios*).

223. Prove that the set of linear transformations which consists of the rotation of a plane round the coordinate origin by angles which are multiples of α, forms a cyclical group. In which case does this group consists of a finite number of transformations?

224. (1) Prove that the set of transformations of the form $w = (az+b)/(cz+d)$ where a, b, c, d are real integers and $ad-bc = 1$, forms a group (this group is said to be *modular*).

(2) Prove that if a, b, c and d are considered as complex integers (that is, numbers of the form $m+ni$, where m and n are real integers), which satisfy the condition $ad-bc = 1$, then the set of transformations of part (1) also forms a group (*Picard's group*).

225. Find the fundamental domains of the groups generated by the transformations:

(1) $T(z) = e^{\frac{2\pi i}{n}} z$ (n a natural number);

(2) $T_1(z) = e^{\frac{2\pi i}{n}} z$, $T_2 = 1/z$;

(3) $T(z) = z+\omega$;

(4) $T_1(z) = z+\omega$, $T_2(z) = -z$;

(5) $T_1(z) = z+\omega_1$, $T_2(z) = z+\omega_2$ $(\mathrm{Im}\,\omega_2/\omega_1 \neq 0)$ (the *doubly periodic* group);

(6) $T_1(z) = z+\omega_1$, $T_2(z) = z+\omega_2$, $T_3(z) = -z$;

(7) $T_1(z) = z+\omega$; $T_2(z) = iz$;

(8) $T_1(z) = z+\omega$, $T_2(z) = e^{2\pi i/3} z$;

(9) $T_1(z) = z+\omega$, $T_2(z) = e^{2\pi i/6} z$.

226. Find the group of linear transformations corresponding in stereographic projection to the rotation of the sphere:

(1) Round the vertical diameter;

(2) Round the diameter parallel to the real axis;

(3) Round the diameter parallel to the imaginary axis;

(4) Round the diameter, the stereographic projection of one of the ends of which is the point a.

HINT. If z_1, z_2 are the images of diametrically opposite points on the sphere, then $z_1\bar{z}_2 = -1$ (see problem **43** of Chapter I).

227. (1) Prove that the group of linear transformations which correspond to a rotation of the sphere, and transform the points with stereographic projections a and b into one another, is defined by the relation

$$\frac{w-b}{1+\bar{b}w} = e^{i\alpha}\frac{z-a}{1+\bar{a}z}.$$

(2) Prove that the differential

$$ds = \frac{|dz|}{1+|z|^2}$$

is invariant with respect to the transformations of this group and represents the spherical length of the element of arc dz (that is, the length of the image of this element on the sphere).

Linear transformations and non-Euclidean geometry

In the realisation of non-Euclidean geometry in the unit circle the part of straight lines is played by arcs of circles in the unit circle and orthogonal to it; the role of motion is played by linear transformations of the unit circle onto itself, the role of distance between the points z_1 and z_2 by the quantity

$$\varrho(z_1, z_2) = \frac{1}{2}\log(\alpha, \beta, z_2, z_1),$$

where α and β are the points of intersection of the "straight lines" passing through the points z_1 and z_2, with the unit circle (the order of the points is

$a, z_1, z_2, \beta)$, and (a, β, z_2, z_1) is the anharmonic ratio of the given points. The angles are measured as in Euclidean geometry (see, for example, [I, Chapter II, § 4, section 8]).

228. Prove that $\varrho(z_1, z_2) > 0$ if $z_1 \neq z_2$ and $\varrho(z, z) = 0$.

229. Prove that $\varrho(z_1, z_3) \leqslant \varrho(z_1, z_2) + \varrho(z_2, z_3)$ where the sign of equality applies only when the point z_3 lies on the "segment", connecting the points z_1 and z_2.

230. Prove that if one of the points z_1 and z_2 tends to a point of the unit circle (or both of them to different points of the unit circle), then the non-Euelidean distance $\varrho(z_1, z_2)$ tends to infinity (that is, the points of the unit circle correspond to the infinitely distant points of the non-Euclidean plane).

231. Prove that the differential

$$ds = \frac{|dz|}{1-|z|^2} \qquad (|z| < 1)$$

is invariant with respect to the group of linear transformations which transforms the circle $|z| < 1$ into itself and represents the non-Euclidean length of the element of arc dz.

HINT. Write down the general form of the transformation of the disk $|z| < 1$ into itself which transforms the point a into the point b ($|a| < 1$, $|b| < 1$).

232. Indicate methods for the construction of the following curves:

(1) The pencil of "straight lines" passing through the point z_0;

(2) "The straight line" passing through the points z_1 and z_2;

(3) The equidistance of a "straight line" (the geometrieal locus of points "equally distant" from the given "straight line");

(4) Limiting curves (curves orthogonal to the pencil of "parallel straight lines").

233. (1) Prove that for the "rectilinear" triangle with angles ϕ_1, ϕ_2, ϕ_3 the inequality $\phi_1+\phi_2+\phi_3 < \pi$ holds.

(2) Prove that the "rectilinear" triangle apart from "motion" is defined by its angles ϕ_1, ϕ_2, ϕ_3. Construct the "rectilinear" triangle from its angles.

§ 3. Rational and algebraic functions

The general mapping of a disk or half-plane onto a simply connected domain of the w-plane is of the form $w = \phi[l(z)]$ where $\phi(z)$ is a particular mapping and l is an arbitrary bilinear mapping of the disk er half-plane onto itself (the inverse mapping is of the form $z = l[\psi(w)]$). It is necessary to keep this

remark in mind when finding a normed mapping, that is, a mapping which satisfies certain supplementary conditions. If the condition of normalisation is not given then one of the mapping functions is usually indicated in the answer.

In the practical construction of conformal mappings an important part is played by certain general principles (see, for example, [2, Chapter IX, § 5 and Chapter X, § 7] or [3, Chapter II, §§ 1 and 3].

The Riemann–Schwarz principle of symmetry

Let the boundary of the domain D_1 contain an arc of the circle C (in particular a rectilinear segment), and let the function $w = f_1(z)$ effect the conformal mapping of this domain onto a domain D_1^* so that the arc C passes again into an arc of a circle or rectilinear segment C^*. Then the function $f_2(z)$ which at points symmetrical with respect to C assumes values symmetrical to the values of $f_1(z)$ with respect to C^*†, will be analytic in the domain D_2 symmetrical to the domain D_1 with respect to C and will map it onto the domain D_2^*, symmetrical to D_1^* with respect to C^*.

The function

$$w = \begin{cases} f_1(z) & \text{in } D_1, \\ f_1(z) = f_2(z) & \text{on } C, \\ f_2(z) & \text{in } D_2 \end{cases}$$

effects the conformal mapping of the domain $D_1 + C + D_2$ onto the domain $D_1^* + C^* + D_2^*$††.

The principle of corresponding boundaries

Let D and D^* be simply connected domains with boundaries C and C^* the domain D^* being situated entirely in the finite part of the plane. If the function $w = f(z)$ is analytic in D, continuous on $\overline{D}$ and maps C onto C^* one-one with preservation of the direction of traversal then it effects the one-one and conformal mapping of the domain D onto D^*.

In the solutions of the problems of this and the following sections in cases when the mapping is effected by a branch of a many-valued function, it is recommended that the correspondence of points on the boundaries of the mapped domain and its image should be traced out (this especially refers to the mapping of domains with cuts).

234. By means of the function $w = z^2$ and its inverse find the conformal mapping of the following domains:

(1) The interior of the right hand branch of the equilateral hyperbola $x^2 - y^2 = a^2$ onto the upper half-plane;

(2) The exterior of the parabola $y^2 = 2px$, $p > 0$ (that is the domain bounded by this parabola and not containing its focus) onto the right hand half-plane.

† If C and C^* are segments of the real axis (it is always possible to arrange this by making an additional bilinear transformation), then $f_2(z) = \overline{f_1(\bar{z})}$.

†† The mapping will be one-one if the domains D_1 and D_2 and also D_1^* and D_2^* do not intersect.

REMARK. For the mapping of domains bounded by curves of the second order, see also problems **264, 265, 291, 292, 293, 328.**

235. By means of the functions indicated in the preceding problem map:

(1) The interior of the circle $r = a\cos\phi$ $(a > 0)$ onto the exterior of the cardioid $\varrho = \frac{1}{2}(1+\cos\theta)$;

(2) The interior of the same circle onto the interior of the right hand branch of the lemniscate $\varrho = \sqrt{(\cos 2\theta)}$;

(3) The disk $|z| < 1$ onto the interior of the cardioid $\varrho = A(1+\cos\theta)$, $A > 0$ so that $w(0) = A/8$, $w'(0) > 0$.

236. Find the domain onto which the disk $|z| < 1$ is mapped by means of the function

$$w = R(z+mz^2), \qquad R > 0, \qquad 0 \leqslant m \leqslant \tfrac{1}{2}.$$

Find the image of the polar net of the z-plane.

237. Find the domain onto which the semicircle $|z| < 1$, $\operatorname{Re} z > 0$ is mapped by the function

$$w = z + z^2.$$

238. (1) Find the domain onto which the disk $|z| < 1$ is mapped by the function

$$w = R\big(z+(z^n/n)\big), \qquad R > 0,$$

where n is an integer, $n > 1$.

(2) Find the domain onto which the exterior of the unit disk $|z| > 1$ is mapped by the function $w = R\big(z+(1/nz^n)\big)$, $R > 0$, n an integer, $n > 1$.

REMARK. For mappings involving the function

$$w = R(z+(1/z))$$

(Zhukovskii's function) see problem **261** *et seq.*

239. (1) Explain for what values of m the function $w = R(z+mz^n)$, where n is a natural number, effects the conformal mapping of the disk $|z|<1$ onto a certain domain, and find this domain.

(2) Explain the same questions for the mapping of the exterior of the disk $|z|<1$ by means of the function $w = R\big(z+(m/z^n)\big)$ and of the interior of the same disk by the function $w = R\big((1/z)+mz^n\big)$.

---

240. (1) Map the angle $0 < \arg z < \pi\alpha$, $(0 < \alpha \leqslant 2)$ onto the upper half-plane.

(2) Map the angle $-\pi/4 < \arg z < \pi/2$ onto the upper half-plane so that $w(1-i) = 2$, $w(i) = -1$, $w(0) = 0$.

241. Find the function $w(z)$, which maps the semicircle $|z|<1$, $\operatorname{Im} z>0$ onto the upper half-plane with the conditions:

(1) $w(-1)=0, \quad w(0)=1, \quad w(1)=\infty$;

(2) $w(\pm 1)=\mp 1, \quad w(0)=\infty$;

(3) $w(i/2)=i, \quad \arg w'(i/2)=-\pi/2$.

242. Find the function $w(z)$, which maps the semicircle $|z|<1$, $\operatorname{Im} z>0$ onto the disk $|w|<1$ with the conditions:

(1) $w(\pm 1)=\pm 1, \quad w(0)=-i$;

(2) $w(i/2)=0, \quad \arg w'(i/2)=\pi/2$.

243. Find the function $w(z)$, which maps the domain $|z|>1$, $\operatorname{Im} z>0$ onto the upper half-plane.

244. Map onto the upper half-plane:

(1) The sector $|z|<R, \quad 0<\arg z<\pi\alpha \ (0<\alpha\leqslant 2)$;

(2) The domain $|z|>R, \quad 0<\arg z<\pi\alpha \ (0<\alpha\leqslant 2)$.

In problems **245–250** map onto the upper half-plane the circular lunes (two-angles).

245. $|z|<1, \quad |z-i|<1$.

246. $|z|<1, \quad |z-i|>1$.

247. $|z|>1, \quad |z-i|<1$.

248. $|z|>1, \quad |z-i|>1$.

249. $|z|>2, \quad |z-\sqrt{2}|<\sqrt{2}$.

250. The exterior of the upper unit semicircle.

In problems **251–259** map the given domains onto the upper half-plane.

251. The plane with a cut along the segment $[-1, 1]$.

252. The plane with a cut along the segment $[-i, i]$.

253. The plane with a cut along the segment $[z_1, z_2]$.

254. The plane with cuts along the rays $(-\infty, -R]$, $[R, \infty)$ $(R>0)$.

255. The plane with a cut along the ray in the first quadrant which issues from the point i and is parallel to the straight line $y=x$.

256. The half-plane $\operatorname{Im} z>0$ with a cut along the segment $[0, ih]$, $h>0$.

257. The half-plane $\operatorname{Im} z>0$ with a cut from ih to ∞ along the positive imaginary semi-axis $(h>0)$.

258. The disk $|z|<1$ with a cut along the radius $[0, 1]$.

259. The exterior of the unit circle with a cut along the ray $[1, \infty)$.

260. Find the mapping of the disk $|z|<1$ onto the w-plane with a cut along the ray $(-\infty, -1/4]$ subject to the conditions $w(0)=0$, $w'(0)>0$.

261. Find the transformation of the polar net $|z| = R$, $\arg z = \alpha$ by the function $w = \frac{1}{2}\left(z+\frac{1}{z}\right)$.

262. Find the domain onto which the function $w = \frac{1}{2}\left(z+\frac{1}{z}\right)$ maps:

(1) The disk $|z|<R<1$;
(2) The domain $|z|>R>1$;
(3) The disk $|z|<1$;
(4) The domain $|z|>1$;
(5) The half-plane $\operatorname{Im} z>0$;
(6) The half-plane $\operatorname{Im} z<0$;
(7) The semicircle $|z|<1$; $\operatorname{Im} z>0$;
(8) The semicircle $|z|<1$, $\operatorname{Im} z<0$;
(9) The domain $|z|>1$, $\operatorname{Im} z>0$;
(10) The domain $1<|z|<R$, $\operatorname{Im} z>0$;
(11) The domain $R<|z|<1$, $\operatorname{Im} z>0$;
(12) The domain $1/R<|z|<R$, $\operatorname{Im} z>0$; $\operatorname{Re} z>0$;
(13) The angle $\pi/2-\alpha<\arg z<\pi/2+\alpha$ $(0<\alpha<\pi/2)$.

263. Find the transformation of the polar net using the functions:

(1) $w = (z-1/z)/2$;
(2) $w = (z+a^2/z)/2$ $(a > 0)$;
(3) $w = (z+c^2/z)/2$, $c = |c|e^{i\gamma}$ $(0 \leqslant \gamma < \pi)$.

264. Using the function $w = \frac{1}{2}\left(z+\frac{1}{z}\right)$, map:

(1) The exterior of the segment $[-c, c]$, $(c > 0)$ onto the exterior of the unit circle with the condition that $w(\infty) = \infty$, $\arg w'(\infty) = \alpha$;

(2) The exterior of the ellipse $x^2/a^2+y^2/b^2 = 1$ onto the exterior of the unit circle so that $w(\infty) = \infty$, $\arg w'(\infty) = 0$;

(3) The upper half-plane with the semi-ellipse $x^2/a^2+y^2/b^2 \leqslant 1$; $y > 0$ removed, onto the upper half-plane.

265. Map the doubly connected domain bounded by the confocal ellipses

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1, \qquad \frac{x^2}{a^2+k^2}+\frac{y^2}{b^2+k^2} = 1 \quad (a > b)$$

onto a concentric circular ring with centre at the coordinate origin, and find the modulus of the given doubly connected domain (see p. 29).

266. Find the domain onto which the function $w = \frac{1}{2}\left(z+\frac{1}{z}\right)$

maps the disk $|z| < 1$ with a cut along the segment $[a,1]$ $(-1 < a < 1)$. Consider the cases $a > 0$ and $a < 0$.

In problems **267–270** map the given domains onto the upper half-plane.

267. The disk $|z| < 1$ with a cut along the segment $[1/2, 1]$.

268. The disk $|z| < 1$ with cuts along the radius $[-1, 0]$ and the segment $[a,1]$ $(0 < a < 1)$.

269. The upper half of the disk $|z| < 1$ with a cut along the segment $[0, \alpha i]$ $(0 < \alpha < 1)$.

270. The upper half of the disk $|z| < 1$ with a cut along the segment $[\alpha i, i]$ $(0 < \alpha < 1)$.

271. Map the disk $|z| < 1$ with a cut along the segment $[(1-h)e^{i\alpha}, e^{i\alpha}]$ onto the unit disk of the w-plane.

272. Map the disk $|z| < 1$ with a cut along the segment $[a, 1]$, $0 < a < 1$, onto the disk $|w| < 1$ so that $w(0) = 0$, $w'(0) > 0$. Find $w'(0)$ and the length of the arc corresponding to the cut. For what value of a does the cut pass into a semicircle?

HINT. It is an advantage first to map both the given domain and the disk $|w| < 1$ onto the exterior of the segment.

273. Map the disk $|z| < 1$ with cuts along the segments $[a, 1]$, $[-1, -b]$ $(0 < a < 1,\ 0 < b < 1)$ onto the disk $|w| < 1$ so that $w(0) = 0$, $w'(0) > 0$. Determine $w'(0)$ and the length of the arc corresponding to the cuts.

274. Represent Zhukovskii's function in the form

$$\frac{w-1}{w+1} = \left(\frac{z-1}{z+1}\right)^2$$

and find:

(1) The image of the circle C which passes through the points $z = \pm 1$ at an angle α $(-\pi < \alpha < \pi)$ to the real axis at the point 1, and the domain onto which the exterior of this circle is mapped;

(2) The image of the circle C' which passes through the point $z = 1$ at an angle α to the real axis and contains within it the point -1, and also the domain onto which the exterior of this circle is mapped.

275. (1) Find the images of the circles and domains of the z-plane mentioned in problem **274** if the mapping function $w(z)$ is given by the equation

$$\frac{w-1}{w+1} = \left(\frac{z-1}{z+1}\right)^{2-\delta} \qquad (0 < \delta < 2, \quad w > 0 \quad \text{for} \quad z > 1).$$

(2) In this mapping what is the image of the interior of the circle C?

276. Map the exterior of the unit circle $|z| > 1$ onto the w-plane with a cut along the arc $\arg(w-1)/(w+1) = \beta$ $(0 < |\beta| < \pi)$ in such a way that $w(\infty) = \infty$, $\arg w'(\infty) = \alpha$.

In problems **277–280** find the domains obtained from the mapping of the given domains by the functions indicated.

277. The disk $|z| < 1$; $w = z/(z^2+1)$.

278. The semicircle $|z| < 1$, $\operatorname{Im} z > 0$; $w = 1/(z^2+1)$.

279. The angle $0 < \arg z < \pi/n$; $w = 1/2\,(z^n + 1/z^n)$.

280. The sector $-\pi/n < \arg z < \pi/n$, $|z| < 1$;

$$w = \frac{z}{(1+z^n)^{\frac{2}{n}}} \quad \left(w(z) > 0 \quad \text{for} \quad z > 0 \right).$$

HINT. Represent the mapping function in the form $w = F\{f[\phi(z)]\}$, where

$$\phi(t) = t^n, \quad f(t) = \frac{t}{(1+t)^2}, \quad F(t) = \sqrt[n]{t}.$$

---

281. (1) Using the solution of problem **280** and the principle of symmetry find the image of the unit circle in the mapping

$$w = \frac{z}{(1+z^n)^{\frac{2}{n}}}.$$

(2) Find the function which maps the interior (and exterior) of the unit circle onto the exterior of the "star":

$$|w| \leqslant 1, \quad \arg w = \frac{2\pi k}{n} \qquad (k = 0,\ 1,\ 2,\ \ldots,\ n-1).$$

282. Map onto the exterior of the unit circle:

(1) The whole plane with cuts along the segments $[-1, 1]$ and $[-i, i]$ (the outside of a cross);

(2) The whole plane with cuts along the rays $(-\infty, -1]$, $[1, +\infty)$, $(-i\infty, -i]$ and $[i, +i\infty)$.

283. (1)* Using the function of problem **279**, map the sector $|z| < 1$, $0 < \arg z < \pi/n$ (n an integer) onto itself so that the radial segments $|z| \leqslant \alpha$, $\arg z = 0$ and $|z| \leqslant \alpha$, $\arg z = \pi/n$ $(0 < \alpha < 1)$ pass into the corresponding radii.

(2) Map the exterior of the unit circle with cuts along the segments $1 \leqslant |z| \leqslant \alpha$, $\arg z = 2k\pi/n$ $(k = 0, 1, 2, \ldots, n-1)$ onto the exterior of the unit circle.

HINT. In the solution of (2) use the principle of symmetry.

284. Map onto the upper half-plane and onto the exterior of the unit circle the exterior of the cross consisting of the segment $[-a, b]$ of the real axis and the segment $[-ci, ci]$ of the imaginary axis ($a \geqslant 0$, $b \geqslant 0$, $c \geqslant 0$, $a^2 + b^2 + c^2 \neq 0$).

HINT. Find the function which maps the upper half-plane with a cut along the segment $[0, ci]$ onto the upper half-plane and use the principle of symmetry in virtue of which the exterior of the cross is mapped onto the whole plane with a cut along a segment of the real axis.

285. Map the plane with cuts along the ray $[-a, +\infty)$ ($a \geqslant 0$) and along the segment $[-ci, ci]$ ($c > 0$) onto the upper half-plane.

HINT. See the hint to problem **284**.

286. Map onto the exterior of the unit circle the plane with cuts along the negative part of the imaginary axis and along the lower half of the unit circle.

HINT. It is reduced to problem **282**, (1) by a linear transformation.

287. Map the plane with cuts along the segment $[-ai, 0]$ ($a > 1$) and along the lower half of the unit circle (Fig. 2) onto the upper half-plane.

HINT. It is reduced to problem **284** by a linear transformation.

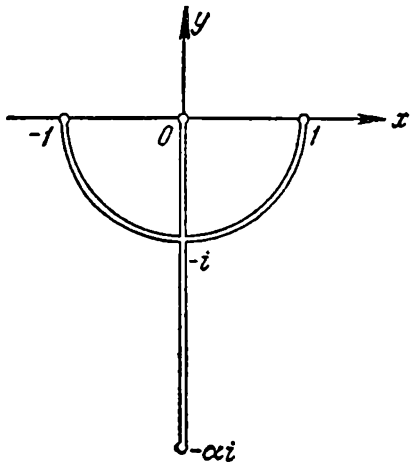

FIG. 2

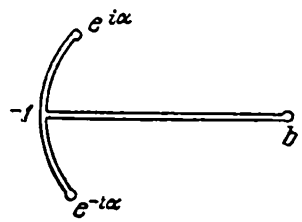

FIG. 3

288. Map onto the upper half-plane the plane with cuts along the segment $[-1, b]$ ($b > -1$) and along the circular arc with ends at the points $c^{\pm i\alpha}$, which passes through the point $z = -1$ (Fig. 3).

289. Map onto the upper half-plane the exterior of the unit circle with cuts along the segments: $[i, bi]$, $[-bi, -i]$, $[1, a]$, $[-a, -1]$ $(a > 1, b > 1)$.

HINT. Zhukovskii's function maps the domain considered onto the domain of problem **284.**

290*. Map onto the exterior of the unit circle the exterior of the "star" represented on Fig. 4.

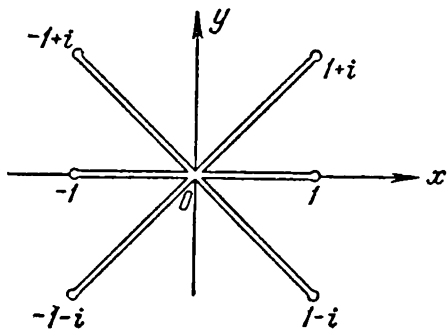

FIG. 4

291*. Map onto the upper half-plane the interior of the right-hand branch of the hyperbola

$$\frac{x^2}{\cos^2\alpha} - \frac{y^2}{\sin^2\alpha} = 1.$$

292*. Map onto the upper half-plane the exterior of the right-hand branch of the hyperbola

$$\frac{x^2}{\cos^2\alpha} - \frac{y^2}{\sin^2\alpha} = 1.$$

293. Map onto the upper half-plane the domain included between the branches of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

In problems **294–295** mappings onto many-sheeted domains (Riemann surfaces) are considered†.

† Here only some of the simplest problems of this type are given. In Chapter IX, § 2 is especially devoted to Riemann surfaces.

294. Find the domains obtained on mapping by means of the function $w = z^2$:

(1) The part of the ring $r_1 < |z| < r_2$, $0 < \arg z < \pi + \alpha$ $(0 < \alpha \leqslant \pi)$;

(2) The domain $|z^2 - 1| < a$ $(0 < a < \infty)$.

295. Find the domains obtained on mapping by means of Zhukovskii's function $w = \frac{1}{2}\left(z + \frac{1}{z}\right)$:

(1) The disk $|z| < R$ $(R > 1)$.

HINT. It is advantageous first to consider the mapping of the disk $|z| < 1$ and the ring $1 < |z| < R$ (see problem **262**).

(2) The disk $\left|\frac{z-1}{z+1}\right| < R$ $(0 < R < \infty)$.

In problems **296–298** construct the Riemann surfaces of the given functions.

296. (1) $w = \sqrt{\left(\frac{z-1}{z+1}\right)}$; (2) $w = \sqrt{(z^2-1)}$.

297. (1) $w = \sqrt{[z(z^2+1)]}$; (2) $w = \sqrt{\left(\frac{z^2+1}{z}\right)}$.

298. $w = \sqrt[3]{(z^2-1)}$.

§ 4. Elementary transcendental functions

299. Explain what the given curves and domains are transformed into by the mapping $w = e^z$:

(1) The rectangular net $x = C$, $y = C$;

(2) The straight lines $y = kx + b$;

(3) The strip $\alpha < y < \beta$ $(0 \leqslant \alpha < \beta \leqslant 2\pi)$;

(4) The strip between the straight lines $y = x$, $y = x + 2\pi$;

(5) The half-strip $x < 0$, $0 < y < \alpha \leqslant 2\pi$;

(6) The half-strip $x > 0$, $0 < y < \alpha \leqslant 2\pi$;

(7) The rectangle $\alpha < x < \beta$, $\gamma < y < \delta$ $(\delta - \gamma \leqslant 2\pi)$.

300. What is the original of the upper half-plane in the mapping $w = (1 + z/n)^n$? What is the limiting image of the upper half-plane as $n \to \infty$?

301. Explain what the following are transformed into by the mapping $w = \log z$:

(1) The polar net $|z| = R$, $\arg z = \theta$;

(2) The logarithmic spirals $r = Ae^{k\phi}$ $(A > 0)$;

(3) The angle $0 < \arg z < \alpha \leqslant 2\pi$;

(4) The sector $|z| < 1$, $0 < \arg z < \alpha \leqslant 2\pi$;

(5) The ring $r_1 < |z| < r_2$, with a cut along the segment $[r_1, r_2]$.

The real and imaginary parts of the function

$$w = \xi + i\eta = \log \frac{a+z}{a-z},$$

are known as the *bipolar coordinates* of the point $z = x + iy$ with respect to the poles $\pm a$ $(a > 0)$.

302. (1) Prove that the function w effects the schlicht mapping of the whole z-plane with the cuts $(-\infty, -a]$ and $[a, \infty)$ onto the strip $-\pi \leqslant \eta \leqslant \pi$ of the w-plane: the upper limits of the cut corresponding to the straight line $\eta = \pi$, and the lower limits to the straight line $\eta = -\pi$ (Fig. 5).

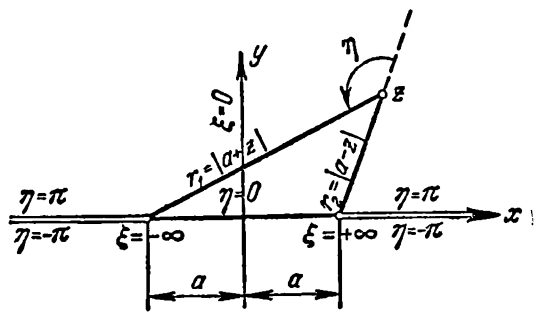

FIG. 5

(2) Establish the correctness of the relations:

$$x = \frac{a \sinh \xi}{\cosh \xi + \cos \eta}, \qquad y = \frac{a \sin \eta}{\cosh \xi + \cos \eta},$$

$$\sqrt{(x^2+y^2)} = r = a \sqrt{\left(\frac{\cosh \xi - \cos \eta}{\cosh \xi + \cos \eta} \right)}.$$

(3) Prove that the originals of the segments $\xi = \xi_0$, $-\pi \leqslant \eta \leqslant \pi$ are the circles of Apollonius with respect to the points $\pm a$:

$$(x - a \coth \xi_0)^2 + y^2 = \left(\frac{a}{\sinh \xi_0} \right)^2$$

(the original of the segment $\xi = 0$, $-\pi \leqslant \eta \leqslant \pi$ is the ordinate axis) (Fig. 6).

(4) Prove that the originals of the lines $\eta = \eta_0$ are arcs of circles passing through the points $\pm a$,

$$x^2+(y+a\cot\eta_0)^2 = \left(\frac{a}{\sin\eta_0}\right)^2,$$

which lie in the upper half-plane for $\eta_0 > 0$ and in the lower half-plane for $\eta_0 < 0$. The line $\eta = 0$ corresponds to the segment $[-a, a]$. The arcs corresponding to the values $\eta = \eta_0$ and $\eta = \eta_0 - \pi$ ($\eta_0 > 0$), supplement one another and make up a complete circle (Fig. 7).

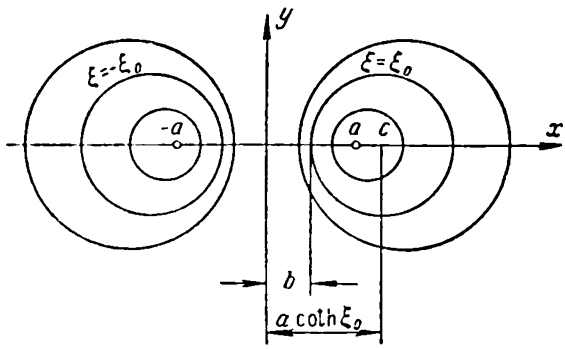

FIG. 6

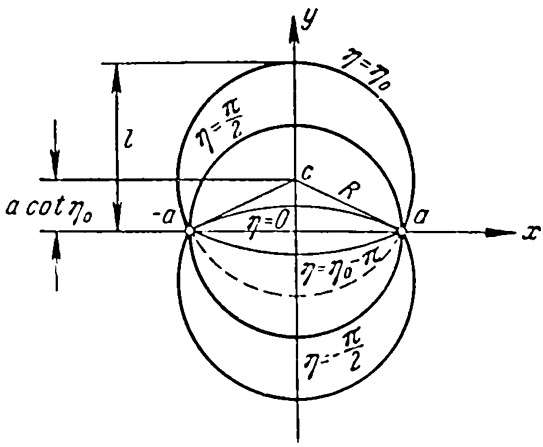

FIG. 7

(5) Find the magnitude of the segments b (see Fig. 6) and l (see Fig. 7).

REMARK. A coordinate net constructed in this way in the z-plane is called a *bipolar net*.

303. Explain what the mapping $w = \cos z$ transforms the following into:

(1) The rectangular net $x = C$, $y = C$;
(2) The half-strip $0 < x < \pi$, $y < 0$;
(3) The half-strip $0 < x < \pi/2$, $y > 0$;
(4) The half-strip $-\pi/2 < x < \pi/2$, $y > 0$;
(5) The strip $0 < x < \pi$;
(6) The rectangle $0 < x < \pi$, $-h < y < h$, $(h > 0)$.

304. Explain what the mapping $w = \sin^{-1} z$ transforms the following into;

(1) The upper half-plane;
(2) The plane with cuts along the real axis along the segments $(-\infty, -1]$, $[1, \infty)$;
(3) The first quadrant;
(4) The half-plane $x < 0$ with a cut along the real axis along the segment $(-\infty, -1]$.

305. Explain what the mapping $w = \cosh z$ transforms the following into:

(1) The rectangular net $x = C$, $y = \mathrm{C}$;
(2) The strip $0 < y < \pi$;
(3) The half-strip $x > 0$, $0 < y < \pi$.

306. Explain what the mapping $w = \sinh^{-1} z$ transforms the following into:

(1) the plane with cuts along the imaginary axis along the rays $1 \leqslant y < \infty$ and $-\infty < y \leqslant -1$;
(2) the first quadrant.

307. Explain what the mapping $w = \tan z$ transforms the following into:

(1) The rectangular net $x = C$, $y = C$;
(2) The half-strip $0 < x < \pi$, $y > 0$;
(3) The strip $0 < x < \pi$;
(4) The strip $0 < x < \pi/4$;
(5) The strip $-\pi/4 < x < \pi/4$.

308. Explain what the mapping $w = \coth z$ transforms the following into:

(1) The strip $0 < y < \pi$;
(2) The half-strip $0 < y < \pi$, $x > 0$.

In problems **309–314** map the given domains onto the upper half-plane.

309. The strip bounded by the straight lines $y = x$, $y = x+h$.

310. The half-strip $x < 1$, $0 < y < h$.

311. The circular lune bounded by the circles $|z| = 2$, $|z-1| = 1$.

312. The domain bounded by the circles $|z| = 2$, $|z-3| = 1$ (the plane with the disks deleted).

313. The domain defined by the inequalities:

$$|z-1| > 1, \quad |z+1| > 1, \quad \operatorname{Im} z > 0$$

(the upper half-plane with two semicircles deleted).

314. The domain included between the confocal parabolas

$$y^2 = 4(x+1), \quad y^2 = 8(x+2).$$

HINT. See problem **234**, (2).

315. Find the function $w(z)$, which maps the domain bounded by the circle $|z| = 1$ and the straight line $\operatorname{Im} z = 1$ (the half-plane $\operatorname{Im} z < 1$ with the disk deleted):

(1) On the disk $|w| < 1$ with the normalisation $w(-3i) = 0$, $\arg w'(-3i) = \pi/3$;

(2) On the disk $|w| < 1$ with the normalisation $w(-3i) = (-1+i)/2$, $\arg w'(-3i) = \pi/2$;

(3) On the upper half-plane with the normalisation $w(-3i) = 1+i$, $\arg w'(-3i) = \pi$.

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316. Map on the upper half-plane:

(1) The strip $0 < x < 1$ with a cut along the line $x = \frac{1}{2}$, $h \leqslant y < \infty$;

(2) The strip $0 < x < 1$ with cuts along the lines $x = \frac{1}{2}$, $h_1 \leqslant y < \infty$; and $x = \frac{1}{2}$, $-\infty < y \leqslant h_2$ $(h_2 < h_1)$.

HINT. First map the strip $0 < x < \frac{1}{2}$ onto the upper half-plane. By the principle of symmetry the mapping function will map the given domain on the whole plane with certain cuts.

In problems **317–327** map the domains indicated onto the upper half-plane.

317. The strip $0 < x < 1$ with a cut along the segment $0 < x < h$, $y = 0$ $(h < 1)$.

318. The strip $0 < x < 1$ with cuts along the segments $0 \leqslant x \leqslant h_1$, $y = 0$ and $1-h_2 \leqslant x \leqslant 1$, $y = 0$ $(h_1+h_2 < 1)$.

319. The half-strip $0 < x < \pi$, $y > 0$ with a cut along the segment $x = \pi/2$, $0 \leqslant y \leqslant h$.

320. The half-strip $0 < x < \pi$, $y > 0$ with a cut along the ray $x = \pi/2$, $h \leqslant y < \infty$ $(h > 0)$.

321. The half-strip $0 < x < \pi$, $y > 0$ with cuts along the segment $x = \pi/2$, $0 \leqslant y \leqslant h_1$ and along the ray $x = \pi/2$, $h_2 \leqslant y < \infty$ $(h_2 > h_1)$.

322. The domain bounded by the circles $|z-1| = 1$, $|z+1| = 1$ with a cut along the ray $2 \leqslant x < \infty$, $y = 0$.

323. The domain bounded by the circles $|z-1| = 1$, $|z-2| = 2$ with a cut along the segment $y - 0$, $2 \leqslant x \leqslant a$ $(a < 4)$.

324. The domain bounded by the circles $|z-1| = 1$, $|z-2| = 2$ with cuts along the segments $y = 0$, $2 \leqslant x \leqslant a$ and $y = 0$, $b \leqslant x \leqslant 4$ $(a < b)$.

325. The domain bounded by the imaginary axis and the circle $|z-1| = 1$, with cuts along the segment $y = 0$, $2 \leqslant x \leqslant a$ and along the ray $y = 0$, $b \leqslant x < \infty$ $(a < b)$.

326. The domain bounded by the circles $|z-1| = 1$, $|z+1| = 1$, with a cut along the segment $x = 0$, $-\alpha \leqslant y \leqslant \beta$ $(\alpha \geqslant 0, \beta \geqslant 0)$.

327. The domain $|z-1| > 1$, $|z+1| > 1$, $\operatorname{Im} z > 0$ (the upper half-plane with semicircles removed) with a cut along the segment $x = 0$, $0 \leqslant y \leqslant h$.

328. Map the interior of the parabola $y^2 = 4\alpha^2(x+\alpha^2)$ onto the upper half-plane and onto the unit disk.

HINT. Make a cut along the axis of symmetry of the parabola, map the upper half of the parabola onto a half-strip (by means of the function $\sqrt{z}$) and then onto the half-plane, and use the symmetry principle.

329*. Map the upper half-plane with cuts along the segments $0 \leqslant y \leqslant a$, $x = \pi/2 + k\pi$ $(k = 0, \pm 1, \pm 2, +\ldots)$ onto the upper half-plane (Fig. 8).

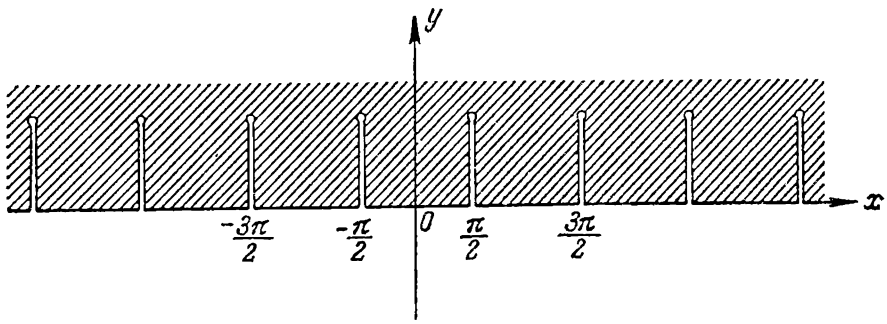

FIG. 8

330. Map the plane with parallel cuts $-a \leqslant x \leqslant a$, $y = \pi/2 + k\pi$ ($k = 0, \pm 1, \pm 2, \ldots$) onto the plane with cuts along the real axis segments $[k\pi - b,\ k\pi + b]$ ($k = 0, \pm 1, \pm 2, \ldots$; $0 < b < \pi/2$).

HINT. Make supplementary cuts along the imaginary axis, map one of the domains formed onto the upper half-plane and use the principle of symmetry.

331. Map the plane with cuts along the rays $(-\infty, -\pi/2]$, $[\pi/2, +\infty)$ and along the segments $-a \leqslant y \leqslant a$, $x = \pi/2 + k\pi$ ($k = 0, \pm 1, \pm 2, \ldots$) onto the exterior of the unit circle (Fig. 9).

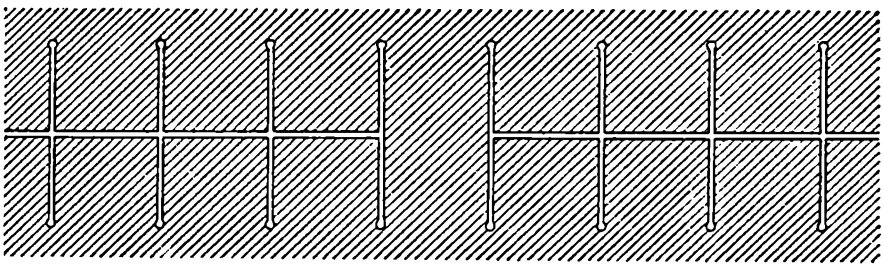

FIG. 9

HINT. Map the function which gives the solution of problem **329** onto the plane with cuts along the rays

$$\left(-\infty, -\frac{1}{\sin^{-1} \operatorname{sech} a}\right], \quad \left[\frac{1}{\sin^{-1} \operatorname{sech} a}, +\infty\right],$$

332. Map the plane with cuts along the rays $(-\infty, p]$, $[q, +\infty)$ ($-\pi/2 \leqslant p < q \leqslant \pi/2$) and along the segments $-a \leqslant y \leqslant a$, $x = \pi/2 + k\pi$ ($k = 0, \pm 1, \pm 2, \ldots$) onto the upper half-plane (Fig. 10).

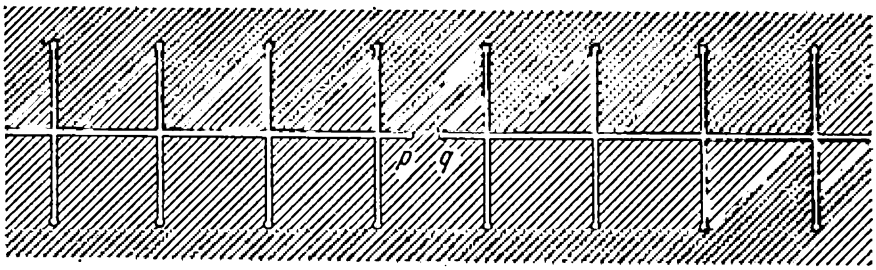

FIG. 10

333*. Map the plane with cuts along the rays $0 \leqslant y < \infty$, $x = k\pi/2$ ($k = 0, \pm 1, \pm 2, \ldots$) onto the upper half-plane.

In problems **334–337** the images are many-sheeted domains (see the footnote on p. 41).

334. Find the domains onto which the following are mapped by means of the function $w = e^z$:

(1) The rectangle $0 < x < a$, $0 < y < b$;

(2) The half-strip $0 < x < a$, $y > 0$;

(3) The strip $0 < x < a$.

335. Find the domains onto which the following are mapped by means of the function $w = \cos z$:

(1) The strip $-\pi/2 < x < \pi/2$;

(2) The strip $0 < x < 2\pi$.

336. Find the domain onto which the strip $0 < x < 2\pi$ is mapped by means of the function $w = \tan z$.

337. Construct the Riemann surface onto which the function $w = e^{\frac{1}{z}}$ maps the z-plane.

§ 5. Boundaries of univalency, convexity and starlikeness

In problems **339–345** r_1 denotes the maximum radius of a circle with centre at the coordinate origin within which the function $w = f(z)$ is univalent; r_2 denotes the maximum radius of a disk with centre at the coordinate origin which the function $w = f(z)$ maps schlicht onto a convex domain and r_3 the maximum radius of a disk with centre at the coordinate origin mapped by the function $w = f(z)$ schlicht onto a domain which is starlike with respect to the point $w = 0$. (A domain is *starlike* with respect to a given point, if any point of the domain can be joined to the given point by a rectilinear segment lying wholly within the domain.) It is obvious that $r_2 \leqslant r_3 \leqslant r_1$.

338. Find r_1, r_2, r_3 for the function $w = z/(1-z)$ and construct the images of the disks $|z| < r_1$, $|z| < r_2$, $|z| < r_3$.

339. Find r_1 for each of the following functions:

(1) $w = z+z^2$;

(2) $w = z+az^2$, (a is a real number);

(3) $w = \dfrac{z}{(1-z)^2}$.

340. Prove that in the mapping $w = f(z)$ the curvature of the image of the circle $|z| = r$ is expressed by the formula

$$k = \frac{1+\operatorname{Re}\left[z\dfrac{f''(z)}{f'(z)}\right]}{|zf'(z)|}.$$

341. Prove that an analytic function $f(z)$ maps the circle $|z| = r$ onto a convex curve when and only when

$$\frac{\partial}{\partial\phi}\left[\frac{\pi}{2}+\phi+\arg f'(z)\right] \equiv 1+\operatorname{Re}\left[\frac{zf''(z)}{f'(z)}\right] \geqslant 0$$

for all ϕ ($z = re^{i\phi}$)†.

342. Prove that the disk $|z| < r$ is mapped by the analytic function $f(z)$ onto a starlike domain when and only when

$$\frac{\partial}{\partial\phi}\arg f(z) = \operatorname{Re}\left[\frac{zf'(z)}{f(z)}\right] \geqslant 0 \text{ for all } \phi \ (z = re^{i\phi}).$$

343. Find r_2 for each of the following functions:

(1) $w = z+z^2$;

(2) $w = z+az^2$ (a is a real number);

(3) $w = z/(1-z)^2$

344. Find r_1 and r_2 for the function $w = e^z-1$.

345. Find r_3 for each of the following functions:

(1) $w = z+z^2$;

(2) $w = z+az^2$ (a is a real number);

(3) $w = z/(1-z)^2$.

HINT. In the solution of problem **345**, (3) it is more convenient to start directly from the inequality $\frac{\partial}{\partial\phi}[\phi+\arg f'(z)] \geqslant 0$.

† See, for example, [4, Chapter XIII, § 2].

CHAPTER III

SUPPLEMENTARY GEOMETRICAL QUESTIONS. GENERALISED ANALYTIC FUNCTIONS

§ 1. Some properties of domains and their boundaries. Mappings of domains

A point of a set E is said to be an *interior* point (of this set) if there exists a neighbourhood of the point which is contained in E. A set E which consists solely of interior points is called an *open* set. An open set is called a *domain* if any two of its points can be joined by a polygonal line all the points of which are contained in the given set. Points which are limit points of an open set E and do not belong to it are called *boundary* points; the set of boundary points forms the boundary Γ of the set E. Points not belonging to either E or Γ are called *exterior* points.

346. (1) Prove that the boundary of a domain is a closed set.

(2) Prove that if the set of exterior points of an open set is not empty then it is open.

(3) Prove that the boundary of a domain is a set which does not possess interior points.

347. (1) Give examples of domains possessing boundary points but not possessing exterior points.

(2) Give examples of simply connected domains for which the set of exterior points is not a domain.

348. The number $\varrho = \inf |z-z_1^*|$, where z_1^* runs through the set E is known as the distance from the point z to the set E. Prove that if E is a closed set a point $z' \in E$ can be found such that $|z-z'| = \varrho$.

349. The number $\varrho = \inf |z_1-z_2|$, where z_1 and z_2 run through the sets E_1, E_2, respectively is known as the distance between these sets. Prove that if E_1 and E_2 are closed and at least one of them is bounded then there exist points $z' \in E_1$ and $z'' \in E_2$ such that $|z'-z''| = \varrho$.

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350. Prove that every closed polygonal curve, that is, a continuous curve consisting of a finite number of segments, can be divided into a finite number of *simple* closed polygonal curves, that is not possessing multiple points.

351. Prove that every polygon can be divided into a finite number of convex polygons.

352. Prove that every simple closed polygonal line divides the points of the plane not belonging to it into two domains, an interior and an exterior, and is identical with the boundary of each of these domains (a particular case of Jordan's theorem, which makes the same assertion for any simple closed continuous curve).

HINT. Let us fix a direction (ray) which is not parallel to any side of the polygon P considered and divide all the points not belonging to P into two classes: we refer to one class those points for which a ray starting from them and parallel to the given direction intersects P in an even number of points $(0, 2, 4, \ldots)$, and to the other class the points for which the same ray gives an odd number of intersections with P. A vertex of P is considered to be a point of intersection if the two sides of P approaching it are situated on different sides of the ray. Let us prove that two points of one class can be connected by a polygonal line which does not have points in common with P, and if two points belong to different classes then any polygonal line connecting them has points in common with P†.

353. Prove that in every simple polygon it is possible to draw a diagonal, all the points of which excluding the ends lie within the polygon.

354*. Prove that a Jordan curve does not contain interior points.

A set $e \subseteq G$ is said to be *open with respect to* G if together with each of its points z_0 it contains all the points $z \in G$ lying in a sufficiently small neighbourhood $U_\varepsilon(z_0)$ of the point z_0 $(|z - z_0| < \varepsilon)$. The set $e \subseteq E$ is said to be *closed with respect to* E if it contains all its limit points belonging to E.

355. Let G be a domain in the z-plane and $E = f(G)$ the image of G in the w-plane in the mapping $w = f(z)$.

Prove the equivalence of the following definitions of the continuity of the mapping:

(1) The mapping $w = f(z)$ of the domain G is continuous at the point $z_0 \in G$ if for every ε-neighbourhood $U_\varepsilon(w_0)$ of the point $w_0 = f(z_0)$ it is possible to find a δ-neighbourhood $U_\delta(z_0)$ of the point z_0, situated in G such that its image belongs to $U_\varepsilon(w_0)$, that is, for $z \in U_\delta(z_0)$ we have $w = f(z) \in U_\varepsilon(w_0)$. The mapping $w = f(z)$ is said to continuous in the domain G, if it is continuous at every point of this domain (Cauchy).

(2) The mapping $w = f(z)$ of the domain G is continuous if the original $f^{-1}(e)$ of every set $e \subset E$, $E = f(G)$, closed with respect to E, is closed in G.

(3) The mapping $w = f(z)$ of the domain G is continuous if the original $f^{-1}(e)$ of every set $e \subset E$, open with respect to E is open in G.

† See R. COURANT and H. ROBBINS, (1942). *What is mathematics?*

(4) The mapping $w = f(z)$ of the domain G is continuous if for every set $g \subseteq G$ the image of the closure of g with respect to G is contained in the closure of $f(g)$ with respect to E.

A mapping (single-valued) $w = f(z)$ is said to be *one-one* or *schlicht*, if two distinct points z always correspond to two distinct points w, that is $f(z_1) \neq f(z_2)$ if $z_1 \neq z_2$. A one-one and mutually continuous mapping is said to be *topological* or *homeomorphic*.

356. Construct a topological mapping of the unit disk $|z| < 1$ onto the whole finite w-plane.

357. Is the following theorem correct? [See, for example, **1**, Chapter I, § 6.1.]

If $w = f(z)$ is a function continuous in the domain G and maps this domain one-one on some set E, then the latter is also a domain, the mapping $w = f(z)$ of the domain G on the domain E being topological (homeomorphic).

Using this theorem prove that if the function $f(z)$ is continuous in the closed region $\overline{G}$ and is one-one within G then the boundary of the domain G is transformed into the boundary of the domain $f(G)$.

A path γ is defined as a continuous image of a directed segment $0 \leqslant t \leqslant 1$, $z = z(t)$ (γ may have multiple points). The point $z(0)$ is said to be the *beginning of the path*, $z(1)$ is its *end*. If $\tau = \phi(t)$ is a homeomorphic mapping of the segment $0 \leqslant t \leqslant 1$ on the segment $0 \leqslant \tau \leqslant 1$ and $\phi(0) = 0$, $\phi(1) = 1$, then the paths $z(t)$ and $z[\phi(\tau)]$ are considered identical. Two paths γ_0, γ_1 with common ends a, b are said to be *continuously deformable into one another* or *homotopic* in the domain G, if there exists in G a family of paths γ_λ $0 \leqslant \lambda \leqslant 1$, with ends a, b, continuously varying with λ (this means that $\max_t |\phi_{\lambda_1}(t) - \phi_{\lambda_2}(t)| \to 0$ as $\lambda_1 \to \lambda_2$) and such that the given paths γ_0, γ_1 correspond to the values $\lambda = 0$, $\lambda = 1$. The closed path γ is said to be *continuously deformable into a point* or *homotopic to zero* in the domain G if it is homotopic in the domain G to a null path, that is, a path consisting of a single point.

358*. Prove the following assertions:

(1) Every path γ in an arbitrary domain is homotopic to a polygonal path.

(2) Every elosed path in a circle is homotopic to zero.

(3) Every closed path in a eircular ring is either homotopic to zero or homotopic to any boundary circle, traversed in a definite direction one or several times.

359. Let γ be an arbitrary path, closed or not. Prove the following assertions:

(1) $\arg(z-a)$, $z \in \gamma$, $a \overline{\in} \gamma$, are uniquely defined as continuous functions of z, if $\arg(z_0-a)$ is given for an arbitrary point $z_0 \in \gamma$.

(2) The quantity $\Delta_\gamma \arg(z-a)$, equal to the increment of $\arg(z-a)$ along γ, does not depend on the choice of z_0 or $\arg(z_0-a)$.

(3) If the path γ lies within an angle with vertex at the point a of magnitude α $(0 < \alpha < 2\pi)$ then $\Delta_\lambda \arg(z-a) < \alpha$.

(4) For a smooth (or even only rectifiable) path γ the equality

$$\Delta_\gamma \arg(z-a) = \int_\gamma \mathrm{d} \arg(z-a)\,.$$

holds.

If γ is a closed path and $a \overline{\in} \gamma$ then the number

$$n_\lambda(a) = \frac{1}{2\pi} \Delta_\gamma \arg(z-a)$$

is said to be the *index* of the point a with respect to γ (or the *winding* number of γ with respect to the point a).

360. Prove the following assertions:

(1) In the homotopic deformation of the path γ taking place outside the point a the index $n_\gamma(a)$ is unchanged.

(2) In every closed path γ it is possible to inscribe a closed polygonal curve π such that $n_\gamma(a) = n_\pi(a)$ (a is a given point outside γ).

(3) If the points a_1, a_2 can be connected by a path which does not encounter the path γ, then $n_\gamma(a_1) = n_\gamma(a_2)$.

(4) If the point a and the path γ are separated by some straight line then $n_\gamma(a) = 0$.

361. Prove that for a simple closed polygonal path π the index $n_\pi(a) = \pm 1$, if a lies within π (the sign depends on the direction of traversal), and $n_\pi(a) = 0$ if a lies outside π.

362*. Let the function $w = f(z)$ effect the topological mapping of the domain G on the domain G'. Let Q be a small circle in G, γ a closed path in Q and a a point of Q which does not belong to γ. Let, finally, γ' and a' be the images of γ and a in G'. Prove that

$$n_{\gamma'}(a') = n_\gamma(a)\,\delta\,,$$

where δ equals either $+1$ or -1, independently of γ and a. In particular prove that if the mapping is continuously differentiable and the Jacobian

$$J = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \neq 0\,,$$

then the sign of δ is identical with the sign of J.

363*. Let $w = f(z)$ effect the continuous mapping of the closed region $\overline{G}$ bounded by the Jordan curve γ. Prove that if $f(z) \neq a$ on γ and for the image $\gamma' = f(\gamma)$ of the curve γ the index $n_{\gamma'}(a) \neq 0$, then within G, $f(z)$ assumes the value a.

364. Prove the equivalence of the following three definitions of a simply connected domain:

(1) The domain G is said to be simply connected if every closed path lying in G is homotopic to zero in this domain.

(2) A domain G which does not contain the point at infinity is simply connected if together with every simple polygon situated in G it also contains its interior.

(3) A domain G different from the extended complex plane is simply connected if it is bounded by a single continuum or point.

REMARK. The extended complex plane is simply connected by the first definition.

§ 2. Quasi-conformal mappings. Generalised analytic functions

If in the relation

$$f(x, y) = f\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) = \phi(z, \bar{z})$$

z and $\bar{z}$ are considered as independent variables the derivatives with respect to these variables are

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

In what follows the notations

$$\frac{\partial w}{\partial z} = w_z, \quad \frac{\partial w}{\partial \bar{z}} = w_{\bar{z}}, \quad \frac{\partial u}{\partial x} = u_x,$$

etc. will be used.

365. Prove the following relations:

(1) $w_z = \frac{1}{2}[(u_x+v_y)+i(-u_y+v_x)]$;

(2) $w_z = \frac{1}{2}[(u_x-v_y)+i(u_y-v_x)]$;

(3) $dw = w_z dz + w_{\bar{z}} d\bar{z}$;

(4) $u_x = \frac{1}{2}(w_z+\overline{w}_{\bar{z}}+w_{\bar{z}}+\overline{w}_z)$;

(5) $u_y = \frac{i}{2}(w_z - \overline{w}_{\bar{z}} - w_{\bar{z}} + \overline{w}_z)$;

(6) $v_x = -\frac{i}{2}(w_z - \overline{w}_{\bar{z}} + w_{\bar{z}} - \overline{w}_z)$;

(7) $v_y = \frac{1}{2}(w_z + \overline{w}_{\bar{z}} - w_{\bar{z}} - \overline{w}_z)$.

366. Prove that the Cauchy–Riemann equations are equivalent to $w_{\bar{z}} = 0$.

367. Prove that Laplace's equation $\Delta u = 0$ can be written in the form $\partial^2 u / \partial z \partial \bar{z} = 0$.

368. Prove that

$$\overline{w}_z = \overline{w_{\bar{z}}}, \quad \overline{w}_{\bar{z}} = \overline{w_z}, \quad \mathrm{d}\overline{w} = \overline{\mathrm{d}w}$$

(the long bar over a symbol indicates passage to the conjugate value after differentiation).

369. Prove that for the function $z(w)$, inverse to $w(z)$,

$$\mathrm{d}z = \frac{\overline{w}_{\bar{z}}}{|w_z|^2 - |w_{\bar{z}}|^2}\mathrm{d}w + \frac{-w_z}{|w_z|^2 + |w_{\bar{z}}|^2}\mathrm{d}\overline{w},$$

370. Prove that the Jacobian of the transformation $w(z)$ is given by

$$J_{w/z} = \frac{D(u, v)}{D(x, y)} = |w_z|^2 - |w_{\bar{z}}|^2.$$

371. Prove the following equalities:

(1) $\frac{\mathrm{d}w}{\mathrm{d}z} = w_z + w_{\bar{z}} \mathrm{e}^{-2i\alpha}$, where $\alpha = \arg \mathrm{d}z$;

(2) $\max \left| \frac{\mathrm{d}w}{\mathrm{d}z} \right| = |w_z| + |w_{\bar{z}}|$;

(3) $\min \left| \frac{\mathrm{d}w}{\mathrm{d}z} \right| = |w_z| - |w_{\bar{z}}|$.

372. Prove that if $\alpha = \arg \mathrm{d}z$, $\alpha' = \arg \mathrm{d}w$, then

(1) $\frac{\mathrm{d}\alpha'}{\mathrm{d}\alpha} = \frac{J_{w/z}}{(u_x \cos\alpha + u_y \sin\alpha)^2 + (v_x \cos\alpha + v_y \sin\alpha)^2}$

$$= \left| \frac{\mathrm{d}z}{\mathrm{d}w} \right|^2 \frac{1}{J_{z/w}};$$

$$(2)\quad \max\frac{d\alpha'}{d\alpha}=p,\qquad \min\frac{d\alpha'}{d\alpha}=\frac{1}{p},$$

$$\text{where}\quad p=\frac{\max\left|\dfrac{dw}{dz}\right|}{\min\left|\dfrac{dw}{dz}\right|}=\frac{|w_z|+|w_{\bar z}|}{\big||w_z|-|w_{\bar z}|\big|}.$$

373. The *set of monogeneity* $\mathfrak{M}_z$ of the function $w=u+iv=f(z)$ at the point z is the set of derived numbers at this point, that is, the set of all possible limiting values of the ratio $\Delta w/\Delta z$ as $\Delta z\to 0$. Prove that if u and v are differentiable at the point z, then $\mathfrak{M}_z$ is either a point or the circumference of a circle.

HINT. Use the relation **371** (1).

374. By the characteristics of an ellipse is meant the ratio p of its semi-axes ($p\geqslant 1$) and, if $p\neq 1$, the angle θ ($0\leqslant\theta<\pi$), formed by the major axis of the ellipse with the axis Ox. Show that the equation of such an ellipse with centre at the coordinate origin and semi-axis minor h can be written in the form

$$\gamma x^2-2\beta xy+\alpha y^2=ph^2$$

where

$$\alpha=p\cos^2\theta+\frac{1}{p}\sin^2\theta,\qquad \beta=\left(p-\frac{1}{p}\right)\cos\theta\,\sin\theta,$$

$$\gamma=p\sin^2\theta+\frac{1}{p}\cos^2\theta\,.$$

REMARK. The quantities α, β, γ are also called characteristics of the ellipse. They are connected by the relation

$$\alpha\gamma-\beta^2=1\,.$$

The function $w=f(z)$ effects an affine (linear) mapping of the z-plane on the w-plane, if

$$u=a_1x+b_1y+c_1,\quad v=a_2x+b_2y+c_2\,,$$

where $\Delta=\begin{vmatrix}a_1 & b_1\\ a_2 & b_2\end{vmatrix}\neq 0\,.$

375. Prove that in an affine transformation parallel straight lines are transformed into parallel straight lines, and families of circles into families of ellipses.

What condition is it necessary to impose on the coefficients of the transformation in order that circles should be transformed into circles (see, for example, [3, Chapter II, § 1])?

376. The characteristics of an affine transformation are the characteristics p, θ or α, β, γ of the ellipses which are transformed into circles. Prove that:

$$(1)\ \frac{\gamma}{a_1^2+a_2^2}=\frac{-\beta}{a_1b_1+a_2b_2}=\frac{\alpha}{b_1^2+b_2^2}=\frac{1}{\Delta};$$

$$(2)\ p=K+\sqrt{(K^2-1)},\quad \text{where } K=\frac{\alpha+\gamma}{2};$$

$$(3)\ \tan\theta=\frac{\gamma-\alpha+\sqrt{[(\gamma-\alpha)^2+4\beta^2]}}{2\beta}.$$

377. By the characteristics of the continuously differentiable mapping $w=u(x,y)+iv(x,y)$ with Jacobian $J>0$ is meant the characteristics $p(z)$, $\theta(z)$ or $\alpha(z)$, $\beta(z)$, $\gamma(z)$ of the affine transformation

$$\mathrm{d}u=u_x\mathrm{d}x+u_y\mathrm{d}y,\quad \mathrm{d}v=v_x\mathrm{d}x+v_y\mathrm{d}y.$$

In this to the infinitely small circle

$$\mathrm{d}u^2+\mathrm{d}v^2=\mathrm{d}\varrho^2$$

there corresponds the infinitely small ellipse

$$\gamma\mathrm{d}x^2-2\beta\mathrm{d}x\mathrm{d}y+\alpha\mathrm{d}y^2=p\mathrm{d}h^2$$

($\mathrm{d}h$ is the minor semi-axis).

Prove the relations:

$$(1)\ \frac{\gamma}{u_x^2+v_x^2}=\frac{-\beta}{u_xu_y+v_xv_y}=\frac{\alpha}{u_y^2+v_y^2}=\frac{p\mathrm{d}h^2}{\mathrm{d}\varrho^2}=\frac{1}{J};$$

$$(2)\ p+\frac{1}{p}=\frac{u_x^2+u_y^2+v_x^2+v_y^2}{J};$$

$$(3)\ p=\frac{\max\left|\frac{\mathrm{d}w}{\mathrm{d}z}\right|}{\min\left|\frac{\mathrm{d}w}{\mathrm{d}z}\right|};$$

$$(4)\ \sqrt{\frac{J}{p}}\leqslant\left|\frac{\mathrm{d}w}{\mathrm{d}z}\right|\leqslant\sqrt{(pJ)}.$$

378. A one-one continuously differentiable mapping $w = u+iv$ with Jacobian $J > 0$ is called a quasi-conformal mapping with characteristics $p(z)$, $\theta(z)$ or $\alpha(z)$, $\beta(z)$, $\gamma(z)$ if it transforms infinitely small ellipses with these characteristics into infinitely small circles. Prove that such a mapping satisfies the system of equations

$$\alpha u_x+\beta u_y = v_y, \quad \beta u_x+\gamma u_y = -v_x,$$

which can also be written in the form

$$w_{\bar{z}} = q(z)w_z, \quad q(z) = -\frac{p-1}{p+1}e^{2i\theta}.$$

HINT. Use the first relation for α, β, γ given in problem **377** and the connection between α, β, γ and p, θ.

379. Prove that a one-one continuously differentiable mapping $w = u+iv$ with positive Jacobian which transforms infinitely small circles into infinitely small ellipses with the characteristics $\alpha(z)$, $\beta(z)$, $\gamma(z)$, satisfies the system of equations

$$u_x+\beta u_y = \alpha v_y, \quad -\beta u_x+u_y = -\alpha v_x,$$

which can be written in thc form

$$w_{\bar{z}} = q(z)\overline{w}_{\bar{z}}, \quad q(z) = \frac{p-1}{p+1}e^{2i\theta}.$$

HINT. Use the relation

$$\gamma\,\mathrm{d}u^2-2\beta\,\mathrm{d}u\,\mathrm{d}v+\alpha\,\mathrm{d}v^2 = J(\mathrm{d}x^2+\mathrm{d}y^2).$$

380. Provc that the equation $w_{\bar{z}} = A(z)w_z$ is invariant with respect to analytic transformations of w, and the equation $w_{\bar{z}} = B(z)\overline{w}_{\bar{z}}$ is invariant with respect to conformal transformations of z.

381. The one-one continuously differentiable mapping $w = u(x, y) +iv(x, y)$ with positive Jacobian is called a quasi-conformal mapping with two pairs of characteristics p, θ; p_1, θ_1 (or with two triads of characteristics α, β, γ; α_1, β_1, γ_1), if it transforms infinitely small ellipses with characteristics p, θ into infinitely small ellipses with characteristics p_1, θ_1.

Prove that sueh a mapping satisfies the system of equations

$$\alpha u_x+(\beta+\beta_1)u_y = \alpha_1 v_y, \quad (\beta-\beta_1)u_x+\gamma u_y = -\alpha_1 v_x,$$

which can also be written in the form

$$w_{\bar{z}} = q_1(z)w_z+q_2(z)\overline{w}_{\bar{z}},$$

where

$$q_1(z) = -\frac{p_1(p^2-1)}{(pp_1+1)(p+p_1)}e^{2i\theta}, \qquad q_2(z) = \frac{p(p_1^2-1)}{(pp_1+1)(p+p_1)}e^{2i\theta_1}.$$

HINT. Use the relation

$$\gamma_1 du^2 - 2\beta_1 du\,dv + \alpha_1 dv^2 = J(\gamma dx^2 - 2\beta dx\,dy + \alpha dy^2).$$

382. Let $p_{w/z}$ be the characteristic p of a quasi-conformal mapping $w(z)$. Prove that

$$p_{w/z} = p_{z/w}$$

and that for the compound quasi-conformal mapping $w[z(t)]$

$$p_{w/t} \leqslant p_{w/z}p_{z/t}$$

383. Show that for the quasi-conformal mappings

$u = f(x),\ v = y$ (longitudinal stretching compression),
$u = x,\ v = f(y)$ (transverse stretching compression),
$\varrho = r,\ \theta = f(\phi)$ (angular stretching compression)

the characteristic is

$$p = \max\left(f', \frac{1}{f'}\right),$$

and for the mapping
$\varrho = f(r),\ \theta = \phi$ (radial stretching compression)

$$p = \max\left(\frac{rf'}{f}, \frac{f}{rf'}\right).$$

384. Construct the quasi-conformal mapping of the disk $|z| < R$ onto itself, transforming the point $z = a$ ($|a| < R$) into the origin and leaving fixed the points of the circumference $|z| = R$. Evaluate the characteristic p.

385. Construct the quasi-conformal mapping of the oblique half-strip $x > 0,\ x\tan\alpha < y < x\tan\alpha + h$ onto the rectangular half-strip $u > 0,\ 0 < v < h$ without stretching on the base and with constant stretching on the lateral side. Evaluate the characteristic p.

386*. Construct the quasi-conformal mapping of the domain, consisting of a half-plane and a circular segment with central angle

$2\beta_0$ (Fig. 11), onto a half-plane with preservation of lengths on the boundary. Evaluate the characteristic p.

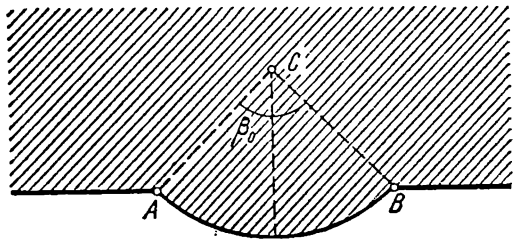

FIG. 11

387. Reduce the quasi-linear equation

$$A\frac{\partial^2 u}{\partial x^2}+2B\frac{\partial^2 u}{\partial x\,\partial y}+C\frac{\partial^2 u}{\partial y^2}=F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)$$

of elliptical type ($AC-B^2>0$) by means of the schlicht mapping

$$\zeta=\zeta(z)=\xi+i\eta$$

to the canonical form

$$\frac{\partial^2 u}{\partial \xi^2}+\frac{\partial^2 u}{\partial \eta^2}=F_1\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right)$$

Prove that the mapping $\zeta(z)$ satisfies Beltrami's system of equations

$$\frac{A\,\partial\xi/\partial x+B\,\partial\xi/\partial y}{\sqrt{(AC-B^2)}}=\frac{\partial\eta}{\partial y}, \qquad \frac{B\,\partial\xi/\partial x+C\,\partial\xi/\partial y}{\sqrt{(AC-B^2)}}=-\frac{\partial\eta}{\partial x}$$

and is a quasi-conformal mapping with the characteristics α, β, γ, determined from the relations

$$\frac{\gamma}{C}=\frac{\beta}{B}=\frac{\alpha}{A}=\frac{1}{\sqrt{(AC-B^2)}}$$

(it is assumed that $A>0$).

A function $w=u+iv$, which satisfies the equation

$$w_{\bar{z}}+Aw+B\bar{w}=F, \tag{1}$$

where A, B and F are functions of z, is known as a *generalised analytic function.*

In problems **388–394** equations and systems of equations are considered which reduce to the form (1) and also some properties of their solutions†.

388. Show that Carleman's system of equations

$$u_x - v_y + au + bv = f,$$

$$u_y - v_x + cu + dv = g,$$

where a, b, c, d, f and g are continuous functions of the variables x and y, can be written in the form (1)

$$w_{\bar{z}} + Aw + B\bar{w} = F.$$

Express A, B and F in terms of the coefficients of the given system.

389. Show that the equation

$$w_{\bar{z}} - q_2(z)\bar{w}_{\bar{z}} + Aw + B\bar{w} = F$$

can be reduced to the form (1) by means of the "affine" transformation

$$w = a(z)\omega + b(z)\bar{\omega}. \tag{2}$$

Find the general form of the transformation (2).

390. Show that the equation

$$w_{\bar{z}} - q_1(z)w_z - q_2(z)\bar{w}_{\bar{z}} + Aw + B\bar{w} = F$$

can be reduced to the form

$$\omega_{\bar{z}} - q_1'\omega_z + A'\omega + B'\bar{w} = F'$$

by means of the transformation of the preceding problem.

HINT. Apply the transformation considered to the given equation and to the equation

$$\bar{w}_z - \overline{q_1(z)}\bar{w}_z - \overline{q_2(z)}w_z + \bar{A}\bar{w} + \bar{B}w = \bar{F};$$

eliminate $\bar{\omega}_z$ and then select the coefficients $a(z)$ and $b(z)$.

391. Show that the equation

$$w_{\bar{z}} - q_1(z)w_z - q_2(z)\bar{w}_{\bar{z}} + Aw + B\bar{w} = F$$

† On this group of problems see the monograph: I. N. VEKUA, *Generalized analytic functions* (*Obobshchennyye analiticheskiye funktsyi*), Chap. III. Fizmatgiz (1958), Moscow, English translation published by Pergamon Press (1962).

can be reduced to the form

$$w_{\bar{\zeta}} - q_2^* \overline{w}_{\bar{\zeta}} + Aw + B\overline{w} = F$$

by changing the independent variable z to the variable ζ connected with z by the relation $\zeta_{\bar{z}} = q_1^* \zeta_z$. Find q_1^* and q_2^* and explain the geometrical meaning of the transformation $\zeta(z)$.

392. Prove that an elliptic system of differential equations of the form

$$\left.\begin{aligned} \alpha_1 v_y &= \alpha u_x + (\beta + \beta_1) u_y + au + bv + f, \\ -\alpha_1 v_x &= (\beta - \beta_1) u_x + \gamma u_y + cu + dv + g \end{aligned}\right\}$$

(the condition of ellipticity is here $\alpha\beta - \gamma^2 > 0$; in addition $\alpha > 0$) can be reduced to the form $w_{\bar{z}} - q_1(z) w_z - q_2(z) \overline{w}_{\bar{z}} + Aw + B\overline{w} = F$, where $|q_1(z)| + |q_2(z)| < 1$, if $\alpha_1 > 0$, and $\big||q_1(z)| - |q_2(z)|\big| > 1$ if $\alpha_1 < 0$.

HINT. See problem **381**. The case $\alpha_1 < 0$ is reduced to the case $\alpha_1 > 0$ by the replacement of $w = u + iv$ by $\overline{w} = u - iv$.

393. Prove that if $w(z)$ is a continuous differentiable solution of the equation

$$w_{\bar{z}} - q_2(z) \overline{w}_{\bar{z}} = 0,$$

where $q_2(z)$ is an analytic function of z and $|q_2(z)| \neq 1$ then

$$w(z) = \frac{\phi(z) + q_2(z)\,\overline{\phi}(z)}{1 - |q_2(z)|^2},$$

where $\phi(z)$ is an arbitrary analytic function.

394. Prove that if $w(z)$ is a twice continuously differentiable solution of the equation

$$w_{\bar{z}} - q_2(\bar{z}) \overline{w}_{\bar{z}} = 0$$

where $q_2(\bar{z})$ is an analytic function of $\bar{z}$ and $|q_2(\bar{z})| \neq 1$, then

$$w(z) = \phi(z) + \int q_2(\bar{z})\, \overline{\phi}_{\bar{z}}\, d\bar{z},$$

where $\phi(z)$ is an arbitrary analytic function of z.

HINT. It is first necessary to prove that $w(z)$ is the sum of an analytic function of z and an analytic function of $\bar{z}$.

CHAPTER IV

INTEGRALS AND POWER SERIES

In the problems of this and also of the following chapters if nothing is said to the contrary, simple closed contours (that is, those without points of self intersection) are traversed in the positive direction.

§ 1. The integration of functions of a complex variable

395. By direct summation prove the equalities:

$$(1)\ \int_{z_0}^{z_1} dz = z_1 - z_0; \quad (2)\ \int_{z_0}^{z_1} z\,dz = \frac{1}{2}(z_1^2 - z_0^2).$$

396. Let C be a simple closed contour bounding an area S. Prove the following equalities:

$$(1)\ \int_C x\,dz = iS; \quad (2)\ \int_C y\,dz = -S; \quad (3)\ \int_C \bar{z}\,dz = 2iS.$$

397. Evaluate the integrals

$$I_1 = \int x\,dz, \quad I_2 = \int y\,dz$$

along the following paths:

(1) Along the radius vector of the point $z = 2+i$;

(2) Along the semicircle $|z| = 1$, $0 \leqslant \arg z \leqslant \pi$ (the commencement of the path is at the point $z = 1$);

(3) Along the circle $|z - a| = R$.

398. Evaluate the integral $\int |z|\,dz$ along the following paths:

(1) Along the radius vector of the point $z = 2-i$;

(2) Along the semicircle $|z| = 1$, $0 \leqslant \arg z \leqslant \pi$ (the commencement of the path is at the point $z = 1$);

(3) Along the semicircle $|z| = 1$, $-\pi/2 \leqslant \arg z \leqslant \pi/2$ (the commencement of the path is at the point $z = -i$);

(4) Along the circle $|z| = R$.

399. Evaluate the integral $\int_C |z|\bar{z}\,dz$, where C is the closed contour consisting of the upper semicircle $|z| = 1$ and the segment $-1 \leqslant x \leqslant 1$, $y = 0$.

400. Evaluate the integral $\int_C \frac{z}{\bar{z}}\,dz$, where C is the boundary of the half ring represented on Fig. 12.

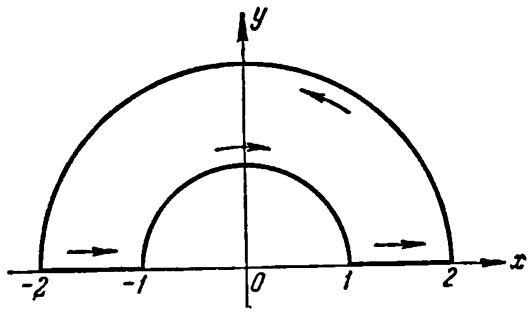

FIG. 12

401. Evaluate the integral $\int (z-a)^n dz$ (n is an integer):

(1) Along the semicircle $|z-a| = R$, $0 \leqslant \arg(z-a) \leqslant \pi$ (the commencement of the path is at the point $z = a+R$);

(2) Along the circle $|z-a| = R$;

(3) Along the perimeter of the square with centre at the point a and sides parallel to the coordinate axes.

In problems **402–405** the branch of the many-valued function which occurs as the integrand is selected by the specification of its value at some point of the contour of integration. If the contour is closed the initial point of the path of integration is always considered to be that point at which the value of the integrand is specified (it must be remembered that the value of the integral may depend on the choice of this initial point).

402. Evaluate the integral $\int \frac{dz}{\sqrt{z}}$ along the following contours:

(1) Along the semicircle $|z| = 1$, $y \geqslant 0$, $\sqrt{1} = 1$;
(2) Along the semicircle $|z| = 1$, $y \geqslant 0$, $\sqrt{1} = -1$;
(3) Along the semicircle $|z| = 1$, $y \leqslant 0$, $\sqrt{1} = 1$;
(4) Along the circle $|z| = 1$, $\sqrt{1} = 1$;
(5) Along the circle $|z| = 1$, $\sqrt{(-1)} = i$.

403. Evaluate the integral $\int_C \operatorname{Log} z\,dz$, where:

(1) C is the unit circle and $\operatorname{Log} 1 = 0$;
(2) C is the unit circle and $\operatorname{Log} i = \pi i/2$;
(3) C is the circle $|z| = R$ and $\operatorname{Log} R = \log R$;
(4) C is the circle $|z| = R$ and $\operatorname{Log} R = \log R + 2\pi i$.

404. Evaluate the integral $\int\limits_{|z|=1} z^n \operatorname{Log} z\, dz$, where n is an integer and:

(1) $\operatorname{Log} 1 = 0$; (2) $\operatorname{Log}(-1) = \pi i$.

405. Evaluate the integral $\int\limits_{|z|=1} z^\alpha\, dz$, where α is an arbitrary complex number and $1^\alpha = 1$.

406. Prove that for any choice of the initial value of the function a^z

$$\int\limits_{|z|=1} a^z dz = 0\,.$$

407. For what values of α $(0 \leqslant \alpha < 2\pi)$ do the following integrals exist:

(1) $I_1 = \int e^{-\frac{1}{z}} dz$; (2) $I_p = \int e^{-\frac{1}{z^p}} dz$

(p is a natural number), taken along the radius vector of the point $z = e^{i\alpha}$?

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408. Prove that if $|a| \neq R$, then

$$\int\limits_{|z|=R} \frac{|dz|}{|z-a||z+a|} < \frac{2\pi R}{|R^2-|a|^2|}\,.$$

409. Prove the following assertions:

(1) If $f(z)$ is continuous in the neighbourhood of the coordinate origin then

$$\lim_{r\to 0} \int\limits_0^{2\pi} f(re^{i\phi})\, d\phi = 2\pi f(0)\,.$$

(2) If $f(z)$ is continuous in the neighbourhood of the point $z = a$ then

$$\lim_{r\to 0} \int\limits_{|z-a|=r} \frac{f(z)\, dz}{z-a} = 2\pi i f(a)\,.$$

410. Prove the following assertions:

(1) If $f(z)$ is continuous in the half-strip $x \geqslant x_0$, $0 \leqslant y \leqslant h$ and the limit $\lim\limits_{x\to\infty} f(x+iy) = A$, not depending on y exists, then

$$\lim_{x\to\infty} \int_{\beta_x} f(z)\, dz = iAh\,,$$

where β_x is a segment of the vertical straight line $0 \leqslant y \leqslant h$, traversed from below upwards.

(2) If $f(z)$ is continuous in the sector $0 < |z-a| \leqslant r_0$, $0 \leqslant \arg(z-a) \leqslant \alpha$, $(0 < \alpha \leqslant 2\pi)$ and the limit

$$\lim_{z\to a}[(z-a)f(z)] = A$$

exists, then

$$\lim_{r\to 0}\int_{\gamma_r} f(z)\,\mathrm{d}z = iA\alpha\,,$$

where γ_r is the arc of the circle $|z-a| < r$ which is present in the given sector, traversed in the positive direction.

(3) If $f(z)$ is continuous in the region $|z| \geqslant R_0$, $0 \leqslant \arg z \leqslant \alpha$ $(0 < \alpha \leqslant 2\pi)$ and the limit

$$\lim_{z\to\infty} zf(z) = A\,,$$

exists, then

$$\lim_{R\to\infty}\int_{\Gamma_R} f(z)\,\mathrm{d}z = iA\alpha\,,$$

where Γ_R is the arc of the circle $|z| = R$, which lies in the given region traversed in the positive direction with respect to the coordinate origin.

411. Prove the following theorems:

(1) If $f(z)$ is continuous in the region $|z| \geqslant R_0$, $\operatorname{Im} z \geqslant a$ (a is a fixed real number) and $\lim_{z\to\infty} f(z) = 0$, then for any positive number m

$$\lim_{R\to\infty}\int_{\Gamma_R} \mathrm{e}^{imz} f(z)\,\mathrm{d}z = 0\,,$$

where Γ_R is the arc of the circle $|z| = R$, which lies in the region considered (Jordan's lemma).

HINT. In the evaluation of the modulus of the integral along the semicircle $|z| = R$, $\operatorname{Im} z > 0$ make use of the inequality $\sin\theta \geqslant 2\theta/\pi$ for $0 \leqslant \theta \leqslant \pi/2$, and in the evaluation along the arcs lying in the lower half-plane (in the case $a < 0$), use the fact that the length of each of them tends to $|a|$ as $R \to \infty$.

(2) If $f(z)$ is continuous in the half-plane $\operatorname{Re} z \geqslant \sigma$ (σ is a fixed real number) and $\lim_{z\to\infty} f(z) = 0$, then for any negative number t

$$\lim_{R\to\infty}\int_{\Gamma_R} \mathrm{e}^{zt} f(z)\,\mathrm{d}z = 0\,,$$

where Γ_R is the arc of the circle $|z| = R$, $\operatorname{Re} z \geqslant \sigma$. If $f(z)$ is continuous in the half plane $\operatorname{Re} z \leqslant \sigma$, the assertion is true if t is positive and Γ_R is the circular arc $|z| = R$, $\operatorname{Re} z \leqslant \sigma$.

REMARK. The proof of both theorems is given, for example, in [3, Chapter V, 2, § 73].

§ 2. Cauchy's integral theorem†

412. Show that if the path does not pass through the coordinate origin, then

$$\int_1^z \frac{d\zeta}{\zeta} = \ln r + i\phi + 2\pi ik$$

where k is an integer indicating how many times the path of integration has encircled the coordinate origin ($z = re^{i\phi}$).

413. Show that if the path does not pass through the points $\pm i$, then

$$\int_0^1 \frac{d\zeta}{1+\zeta^2} = \frac{\pi}{4} + k\pi$$

where k is an integer.

414. Show that if C is an arbitrary simple closed contour which does not pass through the point a, and n is an integer, then

$$\int_C (z-a)^n dz = \begin{cases} 0, & \text{if } n \neq -1, \\ 2\pi i, & \text{if } n = -1, \ a \text{ is inside } C, \\ 0, & \text{if } n = -1, \ a \text{ is outside } C. \end{cases}$$

415. Cauchy's theorem holds in the following stronger form: if $f(z)$ is continuous in the closed region G, bounded by a simple rectifiable contour C, and analytic within G, then $\int_C f(z)\,dz = 0$. Prove this for the case of a starlike contour††.

HINT. Assuming C to bo starlike with respect to the coordinate origin consider the contour $C_\lambda : \zeta = \lambda z$ $(0 < \lambda < 1, z \in C)$ and pass to the limit as $\lambda \to 1$ (see, for example, [2, Chapter V, sec. 8] or [3, Chapter I, § 4, sec. 12]).

† The problems on the evaluation of integrals given in this and in the following section are mainly of an illustrative nature. The majority of problems of this type are located in Chapter VII on the calculus of residues.

†† A contour is said to be starlike with respect to some point if every ray issuing from this point meets tho contour in a single point.

416. Prove the following assertions:

(1) If $f(z)$ is analytic in the strip $0 \leqslant y \leqslant h$, $\lim\limits_{x\to\pm\infty} f(x+iy) = 0$ and the integral $\int\limits_{-\infty}^{\infty} f(x)\,\mathrm{d}x$ exists, then the integral $\int\limits_{-\infty}^{\infty} f(x+ih)\,\mathrm{d}x$ also exists and they are both equal.

(2) If $f(z)$ is analytic in the angle $0 \leqslant \arg z \leqslant \alpha$ $(0 < \alpha \leqslant 2\pi)$, $\lim\limits_{z\to\infty} z\,f(z) = 0$ and the integral $\int\limits_{0}^{\infty} f(x)\,\mathrm{d}x$ exists, then the integral $\int f(z)\,\mathrm{d}z$ along the ray $z = r\mathrm{e}^{i\alpha}$, $0 \leqslant r < \infty$, also exists and they are equal to one another.

HINT. Use the results of problem **410**.

417. Prove that

$$\int\limits_{0}^{\infty} \mathrm{e}^{-x^2}\cos 2bx\,\mathrm{d}x = \frac{\sqrt{\pi}}{2}\mathrm{e}^{-b^2}.$$

HINT. Integrate the function $f(z) = \mathrm{e}^{-z^2}$ along the boundary of the rectangle $|x| \leqslant R$, $0 \leqslant y \leqslant b$ and use Laplace's integral

$$\int\limits_{-\infty}^{\infty} \mathrm{e}^{-t^2}\,\mathrm{d}t = \sqrt{\pi}.$$

418. Prove the equalities

$$\int\limits_{0}^{\infty} \cos x^2\,\mathrm{d}x = \int\limits_{0}^{\infty} \sin x^2\,\mathrm{d}x = \frac{\sqrt{\pi}}{2\sqrt{2}} \quad \text{(Fresnel's integrals)}.$$

HINT. Integrate the function $f(z) = \mathrm{e}^{iz^2}$ along the boundary of the sector $0 \leqslant |z| \leqslant R$, $0 \leqslant \arg z \leqslant \frac{\pi}{4}$ and use the result of problem **411** (1) (put $z^2 = \zeta$).

419. Prove that

$$\int\limits_{0}^{\infty} \frac{\sin x}{x}\,\mathrm{d}x = \frac{\pi}{2} \quad \text{(Dirichlet's integral)}.$$

HINT. Integrate the function $f(z) = \mathrm{e}^{iz}/z$ along the boundary of the domain $r \leqslant |z| \leqslant R$, $0 \leqslant \arg z \leqslant \pi$ and use the results of problems **410** and **411**.

420. Prove that the following equations hold for $0 < s < 1$:

$$(1)\ \int_0^\infty x^{s-1}\cos x\,dx = \Gamma(s)\cos\frac{\pi s}{2},$$

$$(2)\ \int_0^\infty x^{s-1}\sin x\,dx = \Gamma(s)\sin\frac{\pi s}{2}.$$

HINT. Integrate the function $f(z) = z^{s-1}e^{-iz}$ along the boundary of the domain $r \leqslant |z| \leqslant R$, $-\frac{\pi}{2} \leqslant \arg z \leqslant 0$; use the results of problems **410** (2) and **411** (1) and the integral representation of the Gamma function:

$$\Gamma(t) = \int_0^\infty x^{t-1}e^{-t}\,dt.$$

§ 3. Cauchy's integral formula

Everywhere in the problems of this section C denotes a simple closed rectifiable contour.

421. Calculate the integral $\int_C \frac{dz}{z^2+9}$, if:

(1) The point $3i$ lies inside the contour C and the point $-3i$ outside it;

(2) The point $-3i$ lies inside the contour C and the point $3i$ outside it;

(3) The points $\pm 3i$ lie inside the contour C.

422. Calculate all the possible values of the integral $\int_C \frac{dz}{z(z^2-1)}$ for different positions of the contour C. It is assumed that the contour C does not pass through any one of the points 0, 1 and -1.

423. How many different values can the integral $\int_C \frac{dz}{\omega_n(z)}$ assume, where $\omega_n(z) = (z-z_1)(z-z_2)\ldots(z-z_n)$ $(z_i \neq z_j)$ and the contour C does not pass through any of the points z_i?

424. Evaluate the integral

$$\int_{|z-a|=a} \frac{z\,dz}{z^4-1}, \qquad a > 1.$$

425. Evaluate the integral $\frac{1}{2\pi i}\int\limits_C \frac{e^z dz}{z^2+a^2}$, if the contour C contains within itself the circle $|z| \leqslant a$.

426. Evaluate the integral $\frac{1}{2\pi i}\int\limits_C \frac{ze^z dz}{(z-a)^3}$, if the point a lies within the contour C.

HINT. Use the formulae for the derivatives of Cauchy's integral.

427. Evaluate the integral $\frac{1}{2\pi i}\int\limits_C \frac{e^z dz}{z(1-z)^3}$ if:

(1) The point 0 lies inside, and the point 1 outside the contour C;

(2) The point 1 lies inside and the point 0 outside the contour C;

(3) The points 0 and 1 both lie inside the contour C.

428. The function $f(z)$ is analytic in the domain bounded by the simple closed contour C, which contains within it the coordinate origin. Prove that for any choice of the branch of $\operatorname{Log} z$

$$\frac{1}{2\pi i}\int\limits_C f'(z)\operatorname{Log} z\, dz = f(z_0)-f(0),$$

where z_0 is the initial point of integration.

HINT. Integrate by parts.

429. Evaluate the integral

$$\frac{1}{2\pi i}\int\limits_C z^2 \operatorname{Log}\frac{z+1}{z-1} dz,$$

if $\operatorname{Log} a = \log a$ for $a > 0$, and for the contour C we have:

(1) The circle $|z| = 2$;

(2) The eircle $|z-1| = 1$ and the initial point of integration is $z = 1 + i$.

430. By Liouville's theorem a function $f(z)$ which is analytic and bounded throughout the whole plane is a constant. Prove this theorem after calculating the integral

$$\int\limits_{|z|=R} \frac{f(z)dz}{(z-a)(z-b)}$$

($|a| < R$, $|b| < R$) and taking its limit as $R \to \infty$.

431. Let $f(z)$ be analytic in the closed domain bounded by the contour C; $z_1, z_2, \ldots, z_n$ are arbitrary distinct points within C and $\omega_n(z) = (z-z_1)(z-z_2)\ldots(z-z_n)$. Show that the integral

$$P(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\omega_n(\zeta)}\,\frac{\omega_n(\zeta)-\omega_n(z)}{\zeta-z}\,d\zeta$$

is a polynomial of degree $(n-1)$ which is equal to $f(z)$ at the points $z_1, z_2, \ldots, z_n$†.

432. Prove the following theorem (Cauchy's formula for an infinite domain):

Let C be a simple closed contour bounding the finite domain D. The function $f(z)$ is analytic in the exterior of the domain D and $\lim_{z\to\infty} f(z) = A$. Then

$$\frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta-z}\,d\zeta = \begin{cases} -f(z)+A, & \text{if the point } z \text{ belongs to the exterior} \\ & \text{of the domain } D, \\ A, & \text{if the point } z \text{ belongs to the domain } D. \end{cases}$$

The contour C is traversed in the positive direction with respect to the domain D.

HINT. First consider the case $A = 0$.

433. Let the function $f(z)$ and the contour C satisfy the conditions of the preceding problem.

Prove that if the coordinate origin belongs to the domain D, then

$$\frac{1}{2\pi i}\int_C \frac{zf(\zeta)}{\zeta z-\zeta^2}\,d\zeta = \begin{cases} 0, & \text{if } z\in D, \\ f(z), & \text{if } z\in D. \end{cases}$$

§ 4. Numerical series

434. Prove that if the series $\sum_{n=1}^{\infty} c_n$ converges and $|\arg c_n| \leqslant \alpha < \pi/2$, then the series converges absolutely.

435. Let the series $\sum_{n=1}^{\infty} c_n$ and $\sum_{n=1}^{\infty} c_n^2$ converge. Prove that if $\operatorname{Re} c_n \geqslant 0$, then the series $\sum_{n=1}^{\infty} |c_n|^2$ also converges.

† The polynomial $P(z)$ is known as Lagrange's interpolation polynomial.

436. The series $\sum_{n=1}^{\infty} c_n$ possesses the property that the four parts of it, each of which consists of the terms contained in one and the same closed quadrant of the plane, converge. Prove that the given series converges absolutely.

437. Prove the formula (Abel's transformation)

$$\sum_{k=m}^{n} a_k b_k = \sum_{k=m}^{n-1} S_k(b_k - b_{k+1}) - S_{m-1}b_m + S_n b_n,$$

where $1 \leqslant m \leqslant n$, $S_k = a_1 + a_2 + \ldots + a_k$ $(k \geqslant 1)$, $S_0 = 0$.

438. Prove that for the convergence of the series $\sum_{n=1}^{\infty} a_n b_n$ where $b_n > 0$, it is sufficient that the partial sums of the series $\sum_{n=1}^{\infty} a_n$ should be bounded and the sequence of numbers $\{b_n\}$ tend monotonically to zero (Dirichlet's rule).

HINT. Use Abel's transformation.

439. Prove that for the convergence of the series $\sum_{n=1}^{\infty} a_n b_n$ where the b_n are real numbers, it is sufficient that the series $\sum_{n=1}^{\infty} a_n$ should converge and the sequence $\{b_n\}$ be monotonic and bounded (Abel's test).

440. Let the sequence of real numbers $\{b_n\}$ satisfy the conditions:

(1) $\lim_{n\to\infty} (\sqrt{n})b_n = 0$; (2) the series $\sum_{n=1}^{\infty} (\sqrt{n})(b_n - b_{n+1})$ converges.

Then, if the sequence $\left\{\frac{S_n}{\sqrt{n}}\right\}$ is bounded $\left(S_n = \sum_{k=1}^{n} a_k\right)$, the series $\sum_{k=1}^{\infty} a_k b_k$ converges.

441. Let $\overline{\lim_{n\to\infty}} \sqrt[n]{|c_n|} = q$. Prove that the series $\sum_{n=1}^{\infty} c_n$ converges (absolutely), if $q < 1$, and diverges, if $q > 1$.

442. By the examples of the series

$$1 + \frac{1}{2^\beta} + \frac{1}{3^\alpha} + \frac{1}{4^\beta} + \ldots \qquad (1 < \alpha < \beta)$$

and

$$\alpha+\beta^2+\alpha^3+\beta^4+\ldots \qquad (0<\alpha<\beta<1),$$

show that a series may converge even when $\overline{\lim}\left|\frac{c_{n+1}}{c_n}\right|>1$.

443. Prove that if $\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|=1$, then for the absolute convergence of the series $\sum_{n=1}^{\infty} c_n$ it is sufficient that $\overline{\lim_{n\to\infty}} n\left(\left|\frac{c_{n+1}}{c_n}\right|-1\right)<-1$ (Raabe's test).

444. Prove Gauss's test: If $\left|\frac{c_{n+1}}{c_n}\right|=1+\frac{a}{n}+O\left(\frac{1}{n}\right)$, where a does not depend on n and $a<-1$, then the series converges absolutely.

In problems **445–454** study the convergence of the series $\sum_{n=1}^{\infty} c_n$.

445. $c_n=n/2^n$.

446. $c_n=n!/n^n$.

447. $c_n=e^{in}$.

448. $c_n=e^{in}/n$.

449. $c_n=e^{in\phi}/n$.

450. $c_n=e^{in}/n^2$.

451. $c_n=e^{\pi i/n}/n$.

452. $$c_n=\frac{\alpha(\alpha+1)\ldots(\alpha+n-1)\beta(\beta+1)\ldots(\beta+n-1)}{n!\gamma(\gamma+1)\ldots(\gamma+n-1)}$$

(the hypergeometric series), $\operatorname{Re}(\alpha+\beta-\gamma)<0$.

453. $c_n=(\cos in)/2^n$.

454. $c_n=(n\sin in)/3^n$.

§ 5. Power series

In problems **455–466** determine the radii of convergence of the series.

455. $\sum_{n=1}^{\infty}\frac{z^n}{n}$.

456. $\sum_{n=0}^{\infty}\frac{z^n}{n!}$.

457. $\sum_{n=1}^{\infty} n^n z^n$.

458. $\sum_{n=1}^{\infty}\frac{n}{2^n}z^n$.

459. $\sum_{n=1}^{\infty} \frac{n!}{n^n} z^n$.

460. $\sum_{n=0}^{\infty} z^{n!}$.

461. $\sum_{n=0}^{\infty} 2^n z^{n!}$.

462. $\sum_{n=0}^{\infty} z^{2^n}$.

463. $\sum_{n=0}^{\infty} [3+(-1)^n]^n z^n$.

464. $\sum_{n=0}^{\infty} (\cos in) z^n$.

465. $\sum_{n=0}^{\infty} [n+a^n] z^n$.

466. $1+\sum_{n=1}^{\infty} \frac{\alpha(\alpha+1)\dots(\alpha+n-1)\beta(\beta+1)\dots(\beta+n-1)}{n!\,\gamma(\gamma+1)\dots(\gamma+n-1)} z^n$.

467. The radius of convergence of the series $\sum_{n=0}^{\infty} c_n z^n$ is equal to R $(0<R<\infty)$. Determine the radii of convergence of the series:

(1) $\sum_{n=0}^{\infty} n^k c_n z^n$;

(2) $\sum_{n=0}^{\infty} (2^n-1) c_n z^n$;

(3) $\sum_{n=0}^{\infty} \frac{c_n}{n!} z^n$;

(4) $\sum_{n=1}^{\infty} n^n c_n z^n$;

(5) $\sum_{n=0}^{\infty} c_n^k z^n$;

(6) $\sum_{n=0}^{\infty} (1+z_0^n) c_n z^n$.

468. The radii of convergence of the series $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} b_n z^n$ are equal respectively to r_1 and r_2. What can be said about the radii of convergence of the series

(1) $\sum_{n=0}^{\infty} (a_n \pm b_n) z^n$; (2) $\sum_{n=0}^{\infty} a_n b_n z^n$; (3) $\sum_{n=0}^{\infty} \frac{a_n}{b_n} z^n$?

469. Sum the following series for $|z| < 1$:

(1) $\sum_{n=1}^{\infty} nz^n$; (2) $\sum_{n=1}^{\infty} \frac{z^n}{n}$; (3) $\sum_{n=0}^{\infty} \frac{z^{2n+1}}{2n+1}$;

(4) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^n}{n}$.

In problems **470–476** investigate the behaviour of the power series on the boundary of the circle of convergence.

470. $\sum_{n=0}^{\infty} z^n$.

471. $\sum_{n=1}^{\infty} z^n/n$.

472. $\sum_{n=1}^{\infty} z^n/n^2$.

473. $\sum_{n=1}^{\infty} (-1)^n z^n/n$.

474. $\sum_{n=1}^{\infty} z^{p^n}/n$ (p is a natural number).

475. $\sum_{n=2}^{\infty} \frac{(-1)^n}{\log n} z^{3n-1}$.

476. $\sum_{n=1}^{\infty} z^{n!}/n^2$.

By Abel's theorem if the series $\sum_{n=0}^{\infty} c_n$ converges then

$$\lim_{r \to 1} \sum_{n=0}^{\infty} c_n r^n = \sum_{n=0}^{\infty} c_n \quad (0 < r < 1).$$

477. Show that the converse to Abel's theorem is not true, that is, give an example of a divergent series $\sum_{n=0}^{\infty} c_n$, for which there exists the limit $\lim_{r \to 1} \sum_{n=0}^{\infty} c_n r^n$.

478. Using Abel's theorem and the solution of problem **469** prove the following equalities:

(1) $\sum_{n=1}^{\infty} \frac{\cos n\phi}{n} = -\log \left| 2 \sin \frac{\phi}{2} \right| \quad (0 < |\phi| \leqslant \pi)$;

(2) $\sum_{n=1}^{\infty} \frac{\sin n\phi}{n} = \frac{\pi-\phi}{2} \qquad (0 < \phi < 2\pi);$

(3) $\sum_{n=0}^{\infty} \frac{\cos(2n+1)\phi}{2n+1} = \frac{1}{2}\log\left|\cot\frac{\phi}{2}\right| \qquad (0 < |\phi| < \pi);$

(4) $\sum_{n=0}^{\infty} \frac{\sin(2n+1)\phi}{2n+1} = \frac{\pi}{4} \qquad (0 < \phi < \pi);$

(5) $\sum_{n=1}^{\infty} (-1)^{n+1}\frac{\cos n\phi}{n} = \log\left(2\cos\frac{\phi}{2}\right) \qquad (-\pi < \phi < \pi);$

(6) $\sum_{n=1}^{\infty} (-1)^{n+1}\frac{\sin n\phi}{n} = \frac{\phi}{2} \qquad (-\pi < \phi < \pi).$

479*. Prove that the series $\sum_{n=1}^{\infty} \frac{(-1)^{[\sqrt{n}]}z^n}{n}$ converges, but conditionally only at all the points of the circle of convergence.

480. Prove that if the sequence of real positive numbers $\{a_n\}$ tends monotonically to zero and the radius of convergence of the series $\sum_{n=0}^{\infty} a_n z^n$ is equal to 1, then this series converges everywhere on the circle $|z| = 1$, excluding, perhaps, the point $z = 1$.

HINT. Use Dirichlet's test of convergence (see problem 438).

481. Prove that if the series $\sum_{n=0}^{\infty} c_n z^n$ converges at the point $\zeta = Re^{i\theta}$ on the circumference of the circle of convergence then it converges uniformly in every closed region G, belonging to the circle of convergence lying in an angle between any two chords of the circle $|z| = R$, issuing from the point ζ, and not containing any points of the circle $|z| = R$ except the point ζ.

REMARK. This assertion is a more general form of Abel's theorem (see, for example, [1, Chapter III, § 7, sec. 3]).

§ 6. The Taylor series

In problems **482–496** expand the given functions in the power series $\sum_{n=0}^{\infty} c_n z^n$ and find the radius of convergence.

482. $\cosh z$.

483. $\sinh z$.

484. $\sin^2 z$.

485. $\cosh^2 z$.

486. $(a+z)^\alpha$ $(a^\alpha = e^{\alpha \log a})$.

487. $\sqrt{(z+i)}$ $\left(\sqrt{i} = \dfrac{1+i}{\sqrt{2}}\right)$.

488. $\dfrac{1}{az+b}$ $(b \neq 0)$.

489. $\dfrac{z}{z^2-4z+13}$.

490. $\dfrac{z^2}{(z+1)^2}$.

491. $\log\dfrac{1+z}{1-z}$.

492. $\tan^{-1} z$ $(\tan^{-1} 0 = 0)$.

493. $\sinh^{-1} z$ $(\sinh^{-1} 0 = 0)$.

494. $\log(z^2-3z+2)$.

495. $\int_0^z e^{z^2} dz$.

496. $\int_0^z \dfrac{\sin z}{z} dz$.

In problems **497–502** expand the given functions in a series of powers of $(z-1)$ and find the radius of convergence.

497. $\dfrac{z}{z+2}$.

498. $\dfrac{z}{z^2-2z+5}$.

499. $\dfrac{z^2}{(z+1)^2}$.

500. $\sqrt[3]{z}$ $\left(\sqrt[3]{1} = \dfrac{-1+i\sqrt{3}}{2}\right)$.

501. $\log z$.

502. $\sin(2z-z^2)$.

In problems **503–507** find the first five terms of the expansion in powers of z of the given functions.

503. $e^{z \sin z}$.

504. $\sqrt{(\cos z)}$ $(\sqrt{(\cos z)} = 1$ when $z = 0)$.

505. $(1+z)^z = e^{z \log(1+z)}$.

506. e^{e^z}.

507. $e^z \log(1+z)$.

508. (1) Expand in a series of powers of z the function $\log(1+e^z)$ (find the recurrence relation between the coefficients of the series).

HINT. First find the expansion of the derivative of the given function.

(2) Prove that the only term of the expansion containing an odd power of z, is $z/2$.

HINT. Use the identity

$$\log(1+e^z)-\log(1+e^{-z}) = z.$$

In problems **509-513** use the multiplication of series and the substitution of a series into a series to expand the given functions in powers of z.

509. $[\log(1-z)]^2$.

510. $[\mathrm{Log}(1-z)]^2 \quad (\mathrm{Log}\, 1 = 2\pi i)$.

511. $(\tan^{-1}z)^2 \quad (\tan^{-1}0 = 0)$.

512. $\tan^{-1}z \log(1+z^2) \quad (\tan^{-1}0 = 0)$.

513. $e^{\frac{z}{1-z}}$.

514. Prove that if the expansion of the function $1/\cos z$ is written in the form

$$\frac{1}{\cos z} = \sum_{n=0}^{\infty} (-1)^n \frac{E_{2n}}{(2n!)} z^{2n},$$

then the numbers E_{2n} (Euler's numbers) satisfy the relations:

$$E_0 = 1, \quad E_0 + \binom{2n}{2} E_2 + \ldots + \binom{2n}{2n} E_{2n} = 0.$$

515. Prove that if the expansion of the function $z/(e^z-1)$ in a series of powers of z is written in the form

$$\frac{z}{e^z-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n,$$

then the numbers B_n (Bernoulli's numbers) satisfy the relations:

$$B_0 = 1, \quad \binom{n+1}{0} B_0 + \binom{n+1}{1} B_1 + \ldots + \binom{n+1}{n} B_n = 0.$$

516. Prove that all the Bernoulli numbers with odd subscripts, except for B_1, are equal to zero.

HINT. Use the identity $\frac{z}{e^z-1} - \frac{(-z)}{e^{-z}-1} = -z$.

517. Expand in a series of powers of z the function $z\cot z$ and find the radius of convergence of the resulting series.

HINT. Use the equality which follows from Euler's formula

$$z\cot z = iz + \frac{2iz}{e^{2iz}-1}.$$

518. Expand the given functions in series of powers of z and find the radii of convergence of the resulting series:

(1) $\log\dfrac{\sin z}{z}$; (2) $\tan z$; (3) $\log\cos z$; (4) $\dfrac{z}{\sin z}$.

519. Prove that the coefficients c_n of the expansion

$$\frac{1}{1-z-z^2} = \sum_{n=0}^{\infty} c_n z^n$$

satisfy the relation $c_n = c_{n-1}+c_{n-2}$ $(n \geqslant 2)$. Find c_n and the radius of convergence of the series.

REMARK. The numbers c_n are known as *Fibonacci numbers*.

520. In the expansion

$$\frac{A+Bz+Cz^2}{\alpha+\beta z+\gamma z^2+\delta z^3} = \sum_{n=0}^{\infty} c_n z^n \qquad (\alpha \neq 0)$$

find c_0, c_1, c_2, and also the recurrence relation between c_n, c_{n-1}, c_{n-2}, c_{n-3} $(n \geqslant 3)$.

If in some circle $|t| < R$ the expansion

$$F(t, z) = \sum_{n=0}^{\infty} f_n(z)\, t^n$$

is satisfied, then the function $F(t, z)$ is said to be a *generating function* for the sequence $\{f_n(z)\}$. It is frequently possible to prove some of the properties of the sequence of functions $\{f_n(z)\}$ from the properties of its generating function.

521. The Bernoulli polynomials $\phi_n(z)$ are defined by the expansion

$$t\frac{e^{tz}-1}{e^t-1} = \sum_{n=1}^{\infty} \frac{\phi_n(z)}{n!} t^n.$$

Prove the following properties of them:

(1) $\phi_n(z+1)-\phi_n(z) = nz^{n-1}$;

(2) if m is a natural number then

$$\frac{\phi_{n+1}(m)}{n+1} = 1+2^n+3^n+ \dots +(m-1)^n;$$

(3) $\phi_n(z) = \sum_{k=0}^{n-1} \binom{n}{k} B_k z^{n-k}$, where B_k are the Bernoulli numbers (see problem **515**).

522. The function $\dfrac{1}{\sqrt{(1-2tz+t^2)}}$ is the generating function of the Legendre polynomials $P_n(z)$:

$$\frac{1}{\sqrt{(1-2tz+t^2)}} = \sum_{n=0}^{\infty} P_n(z)t^n .$$

Prove the relations:

(1) $(n+1)P_{n+1}(z)-(2n+1)zP_n(z)+nP_{n-1}(z) = 0$;

(2) $P_n(z) = P'_{n+1}(z)-2zP'_n(z)+P'_{n-1}(z)$;

(3) $(2n+1)P_n(z) = P'_{n+1}(z)-P'_{n-1}(z)$.

HINT. Differentiate the generating function with respect to t and with respect to z.

523. Using the integral formula for the coefficients of a Taylor series prove that if $-1 < s < 1$ then

$$P_n(s) = \frac{1}{2\pi i}\int_C \frac{\zeta^n \mathrm{d}\zeta}{\sqrt{(1-2s\zeta+\zeta^2)}},$$

where C is a circle of radius $R > 1$ with centre at the point $\zeta = 0$.

524. Prove that the function $(4-t^2)/(4-4tz+t^2)$ is the generating function of the Chebyshev polynomials:

$$T_n(z) = \frac{1}{2^{n-1}}\cos(n\cos^{-1}z).$$

Using the integral formulae for the coefficients of the Taylor series establish that for $n \geqslant 2$

$$4T_{n+1}(z)-4zT_n(z)+T_{n-1}(z) = 0.$$

525. The Hermite-Chebyshev polynomials $H_n(z)$ are defined by the expansion

$$e^{2tz-t^2} = \sum_{n=0}^{\infty} \frac{H_n(z)}{n!} t^n.$$

Prove the following relations:

(1) $H_{n+1}(z) - 2zH_n(z) + 2nH_{n-1}(z) = 0 \quad (n \geqslant 1)$;

(2) $H_n'(z) = 2nH_{n-1}(z) \quad (n \geqslant 1)$;

(3) $H_n''(z) - 2zH_n'(z) + 2nH_n(z) = 0 \quad (n \geqslant 0)$;

(4) $H_n(z) = (-1)^n e^{z^2} \dfrac{d^n(e^{-z^2})}{dz^n}$.

526. The Laguerre polynomials can be defined by the equation

$$L_n(z) = e^z \frac{d^n(z^n e^{-z})}{dz^n}.$$

Find a generating function for the sequence $\{L_n(z)\}$ and by means of it obtain the recurrence formula connecting

$$L_{n-1}(z), \; L_n(z) \quad \text{and} \quad L_{n+1}(z).$$

REMARK. In problems **522–526** only a few of the particular properties of the given systems of polynomials have been considered. For other important properties of them which play an important part in the solution of various problems of mathematical physics, see, for example, [3, Chapter VII, § 2] or R. COURANT and D. HILBERT, *Methods of Mathematical Physics*, (Vol. I, Chapters II and VII, Interscience Publishers, New York).

In problems **527–529** find the solutions of the given differential equations, which satisfy the conditions $w(0) = 0$, $w'(0) = 1$.

527. $w'' - z^2 w = 3z^2 - z^4$.

528. $(1-z^2)w'' - 2zw' + n(n+1)w = 0$.

529. $(1-z^2)w'' - 4zw' - 2w = 0$.

530. Expand the function $\cos(m \sin^{-1} z)$ $(\sin^{-1} 0 = 0)$ in a series of the form $\sum_{n=0}^{\infty} c_n z^n$ after finding a differential equation which is satisfied by this function.

531. The differential equation

$$z(1-z)\frac{d^2w}{dz^2} + [c - (a+b+1)z]\frac{dw}{dz} - abw = 0$$

is known as the *hypergeometric* equation.

Find the solution $w(z)$ of the hypergeometric equation analytic at the point $z = 0$ which satisfies the condition $w(0) = 1$, assuming that c is not equal to zero or a negative integer.

532. Prove that the general solution of the hypergeometric equation is of the form (c not equal to an integer)

$$w = C_1 F(a, b, c, z) + C_2 z^{1-c} F(a+1-c, b+1-c, 2-c, z),$$

where $F(a, b, c, z)$ is the function defined in the preceding problem (the hypergeometric series).

533*. Prove that if c is not equal to zero or a negative integer, then

$$\frac{\mathrm{d}F(a, b, c, z)}{\mathrm{d}z} = \frac{ab}{c} F(a+1, b+1, c+1, z).$$

§ 7. Some applications of Cauchy's integral formula and power series

534. Let the expansion of the function $f(z)$ in the circle $|z| < R$ be of the form

$$f(z) = \sum_{n=0}^{\infty} c_n z^n.$$

(1) Prove that

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\phi})|^2 \mathrm{d}\phi = \sum_{m=0}^{\infty} |c_n|^2 r^{2n} \qquad (r < R).$$

(2) Prove that if

$$\max_{|z|=r} |f(z)| = M(r),$$

then the coefficients c_n satisfy the inequalities (Cauchy's inequalities)

$$|c_n| \leqslant \frac{M(r)}{r^n} \qquad (r < R).$$

(3) Prove that if just one of Cauchy's inequalities becomes an equality, that is, $|c_k| = \frac{M(r)}{r_k}$ then the function $f(z)$ is of the form

$$f(z) = c_k z^k.$$

HINT. Use the following inequality from section 1. $\sum_{n=0}^{\infty} |c_n|^2 r^{2n} \leqslant [M(r)]^2$.

(4) Prove Liouville's theorem†.

535. The function $f(z) = \sum_{n=0}^{\infty} c_n z^n$ is analytic for $|z| \leqslant r$. Prove that the series $\phi(z) = \sum_{n=0}^{\infty} \frac{c_n z^n}{n!}$ converges in the whole plane and the estimates $|\phi(z)| < M e^{\frac{|z|}{r}}$, $|\phi^{(k)}(z)| < \frac{M}{r^k} e^{\frac{|z|}{r}}$ (M is a constant) hold for its sum.

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536. Let the function $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be analytic in the disk $|z| \leqslant 1$ and effect the schlicht mapping of this disk onto the domain G of area S. Prove that

$$S = \pi \sum_{n=1}^{\infty} n |c_n|^2 .$$

HINT. Write the formula for the calculation of the area S in polar coordinates (see problem **152**).

REMARK. If we omit the condition that the mapping is schlicht the separate parts of the domain G must be counted as many times as the corresponding values of the function $f(z)$ are taken in the circle $|z| \leqslant 1$.

537. Prove that if in the conditions of the preceding problem the function $f(z)$ is analytic only in the open disk $|z| < 1$ and if the finite limit $\lim_{r \to 1} S_r = S$ then exists, where S_r is the area of the open circle $|z| \leqslant r < 1$ then the series $\sum_{n=1}^{\infty} n |c_n|^2$ converges and its sum equals S/π. Prove also that if $\lim_{r \to 1} S_r = \infty$, then the series $\sum_{n=1}^{\infty} n |c_n|^2$ diverges.

REMARK. See, for example, [4, Chapter XIII, § 1].

538. (1) Using the solution of problem **536** prove that if $f'(0) = 1$ and the function $f(z)$ maps the disk $|z| \leqslant 1$ conformally and one-one

† See problem **430** where another method of proof of this theorem is given.

onto some domain G, then the area of the domain G is not less than the area of the mapped disk (the extremal property of mapping onto a disk).

(2) Prove that of all the functions $f(z)$ analytic in the disk $|z| \leqslant R$ and satisfying the condition

$$\int_0^{2\pi} |f(Re^{i\phi})|^2 d\phi = M$$

the linear function realises the mapping of the disk onto the domain of least area. Find this area if $f(0) = 0$.

In problems **539–543** use the maximum modulus principle.

539. Prove that if the function $f(z)$ is different from a constant, analytic in the domain G and does not become zero, then the minimum of $|f(z)|$ cannot be attained inside the domain G.

540. Prove that within the domain bounded by a simple closed level line of the modulus of the function $f(z)$ (that is, at all the points of which $|f(z)| = \text{const}$) and contained together with its boundary within the domain of analyticity of the function $f(z)$, there is at least one zero of this funetion ($f(z) \not\equiv C$).

541. Prove that if $P(z)$ is a polynomial of degree n, then the level lines of its modulus $|P(z)| = C$ (lemniscates) can be decomposed into not more than n eonnected components.

542. Prove Schwarz's lemma:
if the function $f(z)$ is analytic in the circle $|z| < 1$, $f(0) = 0$ and $|f(z)| \leqslant 1$, then in the entire circle

$$|f(z)| \leqslant |z| .$$

Prove also that if even at a single interior point of the circle $|f(z)| = |z|$, then $f(z) = e^{i\alpha}z$ (α is a real number).

HINT. Consider the function $f(z)/z$ and apply to it the principle of the maximum modulus.

543. Prove that if in the preceding problem the condition $f(0) = 0$ is replaced by the condition $f(\alpha) = 0$ ($|\alpha| < 1$), then for $|z| \leqslant 1$ the inequality

$$|f(z)| \leqslant \left| \frac{z-\alpha}{1-\bar{\alpha}z} \right|$$

is satisfied.

HINT. Consider the function $\frac{1-\bar{\alpha}z}{z-\alpha}f(z)$.

The point z_0 is said to be a zero of order k of the function $f(z)$, if $f(z_0) = f'(z_0) = \ldots = f^{(k-1)}(z_0) = 0$, $f^{(k)}(z_0) \neq 0$. The point z_0 is said to be an A-point of order k of the function $f(z)$ if it is a zero of order k of the function $f(z) - A$.

544. Prove that the point z_0 is a zero of order k of the analytic function $f(z)$ when and only when the equality

$$f(z) = (z-z_0)^k \phi(z),$$

holds in some neighbourhood of the point z_0, where the function $\phi(z)$ is analytic at the point z_0 and $\phi(z_0) \neq 0$.

545. Find the order of the zero at $z = 0$ for the functions:

(1) $z^2(e^{z^3}-1)$; (2) $6\sin z^3 + z^3(z^6-6)$; (3) $e^{\sin z} - e^{\tan z}$.

546. The point z_0 is a zero of order k of the function $f(z)$ and a zero of order l of the function $\phi(z)$. What is the point z_0 for the following functions:

(1) $f(z)\phi(z)$; (2) $f(z)+\phi(z)$; (3) $f(z)/\phi(z)$?

In problems **547–561** find the order of all the zeros of the given functions:

547. z^2+9.

548. $(z^2+9)/z^4$.

549. $z\sin z$.

550. $(1-e^z)(z^2-4)^3$.

551. $1-\cos z$.

552. $\frac{(z^2-\pi^2)^2 \sin z}{z^7}$.

553. $\frac{1-\cot z}{z}$.

554. $e^{\tan z}$.

555. $\sin^3 z$.

556. $\frac{\sin^3 z}{z}$.

557. $\sin z^3$.

558. $\cos^3 z$.

559. $\cos z^3$.

560. $(\sqrt{z}-2)^3$.

561. $\left(1-\sqrt{(2-2\cos z)}\right)^2$.

562. Prove that the distance of the nearest zero of the function $f(z) = \sum_{n=0}^{\infty} c_n z^n$ to the point $z = 0$ is not less than $\frac{\varrho |c_0|}{M+|c_0|}$, where ϱ is any number not exceeding the radius of convergence of the series, and $M(\varrho) = \max_{|z|=\varrho} |f(z)|$.

HINT. Establish that the function $f(z)$ has no zeros in the domain where $|f(z) - c_0| < |c_0|$: estimate $|f(z) - c_0|$, using Cauchy's inequality (see problem 534).

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563. Does there exist a function analytic at the point $z = 0$ and assuming at the points $z = \frac{1}{n}$ $(n = 1, 2, \ldots)$ the values:

(1) $0, \quad 1, \quad 0, \quad 1, \quad 0, \quad 1, \quad \ldots, \quad 0, \quad 1, \quad \ldots;$

(2) $0, \quad \frac{1}{2}, \quad 0, \quad \frac{1}{4}, \quad 0, \quad \frac{1}{6}, \quad \ldots, \quad 0, \quad \frac{1}{2k}, \quad \ldots;$

(3) $\frac{1}{2}, \quad \frac{1}{2}, \quad \frac{1}{4}, \quad \frac{1}{4}, \quad \frac{1}{6}, \quad \frac{1}{6}, \quad \ldots, \quad \frac{1}{2k}, \quad \frac{1}{2k}, \quad \ldots;$

(4) $\frac{1}{2}, \quad \frac{2}{3}, \quad \frac{3}{4}, \quad \frac{4}{5}, \quad \frac{5}{6}, \quad \frac{6}{7}, \quad \ldots, \quad \frac{n}{n+1}, \quad \ldots?$

564. Do functions exist which are analytic at the point $z = 0$ and satisfy the conditions:

(1) $f\left(\frac{1}{n}\right) = f\left(-\frac{1}{n}\right) = \frac{1}{n^2};$

(2) $f\left(\frac{1}{n}\right) = f\left(-\frac{1}{n}\right) = \frac{1}{n^3}?$

(n a natural number).

565. The function $\sin [1/(1-z)]$ has an infinite sequence of zeros converging to the point $z = 1$, but nevertheless this function is not a constant. Does this contradict the uniqueness theorem?

566. Is it possible for the sequence of zeros (or in general A-points) of a function which is not identically a constant, and is analytic in the whole finite plane, to have a limit point?

567*. The function $f(z) = u(x, y) + iv(x, y)$ is analytic at the point $z_0 = x_0 + iy_0$ and $f(z_0) = c_0$. Prove that

$$f(z) = 2u\left(\frac{z+\bar{z}_0}{2}, \frac{z+\bar{z}_0}{2i}\right) - \bar{c}_0.$$

568. Prove that with the conditions of the preceding problem

$$f(z) = 2iv\left(\frac{z+\bar{z}_0}{2}, \frac{z+\bar{z}_0}{2i}\right)+\bar{c}_0.$$

In problems **569–572** find the analytic function $f(z) = u(x, y) + iv(x, y)$ from its given real or imaginary part.

569. $u(x, y) = x^2-y^2+2$.

570. $u(x, y) = e^x(x\cos y-y\sin y)-\dfrac{y}{x^2+y^2}$.

571. $v(x, y) = x+y-3$.

572. $v(x, y) = \cos x \sinh y-\sinh x \sin y$.

CHAPTER V

LAURENT SERIES, SINGULARITIES OF SINGLE-VALUED FUNCTIONS. INTEGRAL FUNCTIONS

§ 1. Laurent series

In problems **573–588** expand the given function in a Laurent series either in the given ring or in the neighbourhood of the given point. In the latter case determine the domain within which the expansion holds.

573. $1/(z-2)$ in the neighbourhood of the points $z = 0$ and $z = \infty$.

574. $1/(z-a)^k$ ($a \neq 0$, k a natural number) in the neighbourhood of the points $z = 0$ and $z = \infty$.

575. $1/\big(z(1-z)\big)$ in the neighbourhood of the points $z = 0$, $z = 1$, $z = \infty$.

576. $1/\big((z-a)\,(z-b)\big)$ $(0 < |a| < |b|)$ in the neighbourhood of the points $z = 0$, $z = a$, $z = \infty$ and in the ring $|a| < |z| < |b|$.

577. $(z^2-2z+5)/\big((z-2)(z^2+1)\big)$ in the neighbourhood of the point $z = 2$ and in the ring $1 < |z| < 2$.

578. $1/(z^2+1)^2$ in the neighbourhood of the points $z = i$ and $z = \infty$.

579. $\sqrt{\big((z-a)\,(z-b)\big)}$ $(|b| \geqslant |a|)$ in the neighbourhood of the point $z = \infty$ (consider both branches of the function).

580. $f(z) = \sqrt{\left(\dfrac{z}{(z-1)\,(z-2)}\right)}$ $\left(\mathrm{Im} f\left(\tfrac{3}{2}\right) > 0\right)$ in the ring $1<|z|<2$.

581. $z^2 e^{\frac{1}{z}}$ in the neighbourhood of the points $z = 0$ and $z = \infty$.

582. $e^{\frac{1}{1-z}}$ in the neighbourhood of the points $z = 1$ and $z = \infty$.

583. $e^{z+\frac{1}{z}}$ in the domain $0 < |z| < \infty$.

584. $\sin z \sin 1/z$ in the domain $0 < |z| < \infty$.

585. $\sin\big(z/(1-z)\big)$ in the neighbourhood of the points $z = 1$ and $z = \infty$ (in the latter case restriet yourself to the first four terms of the series).

586. $\cot z$ in the neighbourhood of the point $z = 0$ and in the ring $\pi < |z| < 2\pi$.

587. $\log\dfrac{z-a}{z-b}$ in the neighbourhood of the point $z = \infty$.

588. $\dfrac{1}{z-2}\log\dfrac{z-i}{z+i}$ in the neighbourhood of the point $z = \infty$ and in the ring $1 < |z| < 2$.

589. Can the given functions be expanded in a Laurent series in the neighbourhood of the given points:

(1) $\cos 1/z$, $z = 0$;
(2) $\cos 1/z$, $z = \infty$;
(3) $\sec 1/(z-1)$ $z = 1$;
(4) $\cot z$, $z = \infty$;
(5) $\tanh 1/z$, $z = 0$;
(6) $z^2\operatorname{cosec} 1/z$, $z = 0$;
(7) $\log z$, $z = 0$;
(8) $\log 1/(z-1)$, $z = \infty$;
(9) $\log\big((z-1)/(z+i)\big)$, $z = \infty$;
(10) $z^\alpha = (e^{\alpha \log z})$ $z = 0$?

590. Explain whether the given many-valued functions have single-valued branches which can be expanded in a Laurent series (in particular in a Taylor series) in the neighbourhood of the given point:

(1) $\sqrt{(z)}$, $z = 0$;
(2) $\sqrt{\big(z(z-1)\big)}$, $z = \infty$;
(3) $\sqrt{\left(\dfrac{z}{(z-1)(z-2)}\right)}$, $z = \infty$;
(4) $\sqrt[3]{\big((z-1)(z-2)(z-3)\big)}$, $z = \infty$;
(5) $\sqrt[4]{\big(z(z-1)^2\big)}$, $z = \infty$;
(6) $\sqrt{\big(1+\sqrt{(z)}\big)}$, $z = 1$;
(7) $\sqrt{(1+\sqrt{z})}$, $z = 0$;
(8) $\sqrt{\big(z+\sqrt{(z^2-1)}\big)}$, $z = \infty$;
(9) $\sqrt{\big(z+\sqrt{(z^2-1)}\big)}$, $z = 1$;
(10) $\sqrt{\left(1+\sqrt[3]{\left(\dfrac{z}{z+1}\right)}\right)}$, $z = \infty$;
(11) $\operatorname{Log}[(z-1)(z-2)]$, $z = \infty$;
(12) $\operatorname{Log}\dfrac{(z-\alpha)(z-\beta)}{(z-\gamma)(z-\delta)}$, $z = \infty$;
(13) $\sin^{-1} z$, $z = 0$;
(14) $\tan^{-1}(1+z)$, $z = 0$;
(15) $\sinh^{-1}(i+z)$, $z = 0$;
(16) $\sqrt{\left(\dfrac{\pi}{2}-\sin^{-1}z\right)}$, $z = 1$;
(17) $\sqrt{\left(\dfrac{\pi}{4}-\sin^{-1}z\right)}$, $z = \frac{1}{2}\sqrt{2}$.

Some properties of univalent functions†

591. (1) The function $f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$, analytic in the ring $r \leqslant |z| \leqslant R$, effects the schlicht mapping of this ring onto some domain D. Prove that the area S of this domain is equal to

$$S = \pi \sum_{n=-\infty}^{\infty} n |c_n|^2 (R^{2n} - r^{2n}).$$

(2) Prove that the formula for the area S still applies when $f(z)$ is analytic only in the domain $r < |z| < R$; in this case both sides of the equation may tend simultaneously to ∞.

HINT. See problems **536** and **537**.

592. The function $f(z)$ is univalent in the domain $|z| > 1$ and in this domain can be expanded in the Laurent series

$$f(z) = z + \frac{c_{-1}}{z} + \frac{c_{-2}}{z^2} + \dots$$

Prove that

$$\sum_{n=1}^{\infty} n |c_{-n}|^2 \leqslant 1,$$

and explain the geometrical meaning of this inequality (the area theorem).

HINT. Use the fact that the area S_r, bounded by the image of the circle $|z| = r > 1$, is given by ($f(z) = u + iv$)

$$0 \leqslant S_r = \int_{|z|=r} u \, dv = \int_0^{2\pi} \frac{f + \bar{f}}{2} \frac{1}{2i} \left(\frac{\partial f}{\partial \phi} - \frac{\partial \bar{f}}{\partial \phi} \right) d\phi .$$

593. Prove that if the function

$$f(z) = z + c_2 z^2 + \dots$$

is schlicht in the disk $|z| < r$, then $|c_2| \leqslant 2$.

HINT. Prove that the function $\left[F\left(\frac{1}{z}\right) \right]^{-1}$, where $F(z) = \sqrt{(f(z^2))}$ is schlicht in the disk $|z| > 1$, and apply to it the inequality of the preceding problem. (By $\sqrt{(f(z^2))}$ is understood any definite branch of the root.)

† On the subject of this group of problems on the properties of univalent functions see, for example, [1, Chapter V, § 2], where other references to this subject are given.

594. Prove that if under the conditions of the preceding problem $|c_2| = 2$, then the function $f(z)$ is of the form

$$f(z) = \frac{z}{(1+e^{i\alpha}z)^2} \qquad (\alpha \text{ is a real number}).$$

Explain onto what domain the unit disk $|z| < 1$ is mapped by this function.

595*. Prove that if the function $f(z)$ is schlicht in the disk $|z| < 1$ and $f(0) = 0, f'(0) = 1$, then the domain G onto which this function maps the unit disk always contains the circle with centre at the coordinate origin and of radius 1/4 (Köbe's theorem).

596. (1) Prove that if the function $f(z)$ satisfies the conditions of the preceding problem and the points w_1 and w_2 lie on the boundary of the domain G, where $\arg w_2 = \arg w_1 + \pi$, then

$$|w_1| + |w_2| \geqslant 1 \qquad \text{(Szegö's theorem)}.$$

HINT. Apply Köbe's theorem to the function $\dfrac{f(z)}{1-f(z)/w_1}$.

(2) Prove that if the points w_1 and w_2 satisfy the condition $\arg w_2 = \arg w_1 + \alpha (0 \leqslant \alpha \leqslant \pi)$, then

$$|w_1| + |w_2| \geqslant \sin\frac{\alpha}{2}.$$

REMARK. This result strengthens Köbe's theorem only for $\alpha > \pi/3$.

§ 2. Singular points of single-valued analytic functions

In problems **597–630** find the singular points of the functions, explain their nature and investigate the behaviour of the function at infinity†.

597. $\dfrac{1}{z-z^3}$.

598. $\dfrac{z^4}{1+z^4}$.

599. $\dfrac{z^5}{(1-z)^2}$.

600. $\dfrac{1}{z(z^2+4)^2}$.

601. $\dfrac{e^z}{1+z^2}$.

602. $\dfrac{z^2+1}{e^z}$.

† In the answers no distinction is made between a removable and a genuine singularity.

603. ze^{-z}.

604. $\dfrac{1}{e^z-1}-\dfrac{1}{z}$.

605. $\dfrac{e^z}{z(1-e^{-z})}$.

606. $\dfrac{1-e^z}{1+e^z}$.

607. $\tanh z$.

608. $e^{-\frac{1}{z^2}}$.

609. $ze^{\frac{1}{z}}$.

610. $e^{\frac{z}{1-z}}$.

611. $e^{z-\frac{1}{z}}$.

612. $\dfrac{e^{\frac{1}{z-1}}}{e^z-1}$.

613. $\dfrac{1}{\sin z}$.

614. $\dfrac{\cos z}{z^2}$.

615. $\tan z$.

616. $\tan^2 z$.

617. $\dfrac{\cot z}{z^2}$.

618. $\cot z-\dfrac{1}{z}$.

619. $\cot z-\dfrac{2}{z}$.

620. $\dfrac{1}{\sin z-\sin a}$.

621. $\dfrac{1}{\cos z+\cos a}$.

622. $\sin\dfrac{1}{1-z}$.

623. $\cot\dfrac{1}{z}$.

624. $\cot\dfrac{1}{z}-\dfrac{1}{z}$.

625. $\sin\dfrac{1}{z}+\dfrac{1}{z^2}$.

626. $e^{-z}\cos\dfrac{1}{z}$.

627. $e^{\cot\frac{1}{z}}$.

628. $e^{\tan\frac{1}{z}}$.

629. $\sin\left(\dfrac{1}{\sin\dfrac{1}{z}}\right)$.

630. $\sin\left(\dfrac{1}{\cos\dfrac{1}{z}}\right)$.

631. Let $P_n(z)$ and $Q_m(z)$ be polynomials of the nth and mth degree respectively. Describe the behaviour at infinity of the following functions:

(1) $P_n(z)+Q_m(z)$; (2) $P_n(z)/Q_m(z)$; (3) $P_n(z)Q_m(z)$.

632. Prove the equivalence of the following two definitions:

(1) The point x_0 is said to be a pole of order n of the function $f(z)$, if in the Laurent expansion of $f(z)$ in the neighbourhood of z_0

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z-z_0)^n,$$

$c_{-n} \neq 0$, $c_{-(n+1)} = c_{-(n+2)} = \ldots = 0$.

(2) The point z_0 is said to be a pole of order n of the function $f(z)$, if in some neighbourhood of this point $f(z) = \phi(z)/(z-z_0)^n$, where the function $\phi(z)$ is analytic and $\phi(z_0) \neq 0$.

633. Construct examples of functions which have in the extended plane only the following singularities:

(1) A pole of the second order at infinity;

(2) A pole of the second order at the point $z = 0$ with the principal part of the expansion c_{-2}/z^2 and a simple pole at infinity;

(3) Simple poles at the points $z_k = \omega^k$, where $\omega = e^{\frac{2\pi i}{n}}$ ($k = 0, 1, 2, \ldots n-1$).

634. Find the general form of the function which has in the extended plane only the following singularities:

(1) One simple pole;

(2) One pole of order n;

(3) A pole of the second order at the point $z = 0$ with principal part of the expansion $1/z^2$;

(4) A pole of order n at the point $z = 0$ and a pole of order m at infinity;

(5) n poles of the first order.

635. Let $f(z)$ be a single-valued function which has no singularities other than poles in the domain G. Prove that the function $f'(z)/\big(f(z)-A\big)$ (the logarithmic derivative of the function $f(z)-A$) has simple poles at all the poles of the function $f(z)$ and at all the A-points of this function and has no other singular points.

636. What singularity does the function $F(z) = f[\phi(z)]$ have at the point $z = z_0$ (the case $z_0 = \infty$ is included), if the function $\phi(z)$ is either analytic or has a pole at this point, and the point $\zeta_0 = \phi(z_0)$ is a singularity of the following type for the function $f(\zeta)$:

(1) A removable singularity;

(2) A pole of order n;

(3) An essential singularity?

637. The point z_0 (the case $z_0 = \infty$ is included) is an isolated singularity of the function $f(z)$, which maps the circular arc (or rectilinear segment) γ on some circular arc (or rectilinear segment) γ'. What is the nature of the singularity of the function $f(z)$ at the point z_0^*, symmetrical to z_0 with respect to γ (the function $f(z)$ is continued across γ by the principle of symmetry), if the point z_0 is for $f(z)$:

(1) A pole of order n;

(2) An essentially singular point?

638. The Casorati–Weierstrass theorem asserts that if the point z_0 is an essential singularity of the function $f(z)$, then whatever the complex number A may be (including $A = \infty$), a sequence of points $\{z_n\}$ which converges to an essential singularity z_0 exists, such that $\lim_{n\to\infty} f(z_n) = A$. Prove that the theorem remains valid for a non-isolated singularity which is the limit of poles†. (Sometimes singularities of this kind are included among essential singularities.)

639. Find the limits:

$$(1)\ \lim_{y\to\pm\infty} \cot^2 z;\quad (2)\ \lim_{y\to\pm\infty} \frac{1}{\sin z};\quad (3)\ \lim_{x\to\pm\infty} \frac{1}{\cosh z};$$

$$(4)\ \lim_{\substack{x=0\\ y\to 0}} \frac{1}{\sin \frac{1}{z}}.$$

Does the existence of these limits contradict the Casorati–Weierstrass theorem?

Picard's theorem asserts that in the neighbourhood of an essential singularity an analytic function assumes infinitely many times every finite value, with the exception perhaps of one, which is said to be the Picard exceptional value. If meromorphic functions are considered the possible number of exceptional values (including ∞) does not exceed two (see, for example, [1, Chapter VIII, sec. 8.4]).

640. Verify Picard's theorem for the functions:

$$(1)\ e^z;\quad (2)\ e^{\frac{1}{z}};\quad (3)\ \cos\frac{1}{z};\quad (4)\ \tan z;\quad (5)\ \tan^2 z.$$

Find the exceptional values for each of these functions and show that these values (if they exist) are asymptotic, that is, that it is possible to find not more than one line which finishes at an essential singularity along which the function tends to the exceptional value.

† It is assumed that the poles are the only singularities in the neighbourhood of the point considered.

Power series with singularities on the boundary of the circle of convergence†

641. Let the series $f(z) = \sum_{n=0}^{\infty} c_n z^n$ have the radius of convergence R $(0 < R < \infty)$. Let us expand the function $f(z)$ in a series of powers of the difference $(z - re^{i\alpha})$, where $0 < r < R$:

$$f(z) = \sum_{n=0}^{\infty} b_n (z - re^{i\alpha})^n.$$

Prove that if $\overline{\lim_{n\to\infty}} \frac{1}{\sqrt[n]{|b_n|}} = R - r$, then the point $z = Re^{i\alpha}$ is a singularity of the function $f(z)$, and if $\overline{\lim_{n\to\infty}} \frac{1}{\sqrt[n]{|b_n|}} > R - r$ then it is a regular point.

HINT. Use the fact that a power series always has at least one singular point on the boundary of its circle of convergence.

642. Prove Pringsheim's theorem:

If the radius of convergence of the series $\sum_{n=0}^{\infty} c_n z^n = f(z)$ is equal to unity and all the $c_n \geqslant 0$, then the point $z = 1$ is a singularity of the function $f(z)$ (the sum of the series).

HINT. Prove by means of the criterion of the preceding problem that if it is assumed that the point $z = 1$ is regular then all the remaining points of the unit circle will be regular (for the proof use the fact that $|f^{(n)}(z)| \leqslant |f^{(n)}(x)|$, where $x > 0$ and $|z| = x$).

643. Prove that if the radius of convergence of the series $\sum_{n=0}^{\infty} c_n z^n = f(z)$ is equal to unity, where all the c_n are real and the sequence $\{s_n\}\left(s_n = \sum_{k=0}^{n} c_k\right)$ tends to an infinity of a definite sign as $n \to \infty$ (that is beginning with a certain n, all the s_n have the same sign), then the point $z = 1$ will be a singular point for the function $f(z)$.

Prove, for example, that the assertion may not hold if we only have $\lim_{n\to\infty} |s_n| = \infty$.

† See [1, Chapter III, § 6]. See also G. PÓLYA and G. SZEGÖ (1957), *Aufgaben und Lehrsätze aus der Analysis*, 2nd ed. Chapter I, sec. 3, Chapter 5, § 3, or E. C. TITCHMARSH (1951), *Theory of functions*, Chapter VII.

HINT. Estimate $|f(z)|$ for z real and close to 1 ($z < 1$), by means of the identity

$$\sum_{n=0}^{\infty} c_n z^n = (1-z) \sum_{n=0}^{\infty} s_n z^n.$$

644. Prove that if on the circumference of the circle of convergence of the series $\sum_{n=0}^{\infty} c_n z^n = f(z)$ there is just one pole of the function $f(z)$, then the series diverges at all the points of this circumference.

HINT. Using the fact that from the convergence of the series at just one point of the circumference of the circle of convergence it follows that $\lim_{n\to\infty} c_n R^n = 0$ (R is the radius of convergence), prove that if $|z'| = R$ then the radial limit $\lim_{z\to z'} (z'-z)f(z) = 0$.

645. Prove that if the power series $f(z) = \sum_{n=0}^{\infty} c_n z^n$ has on the circumference of the circle of convergence only one singular point z_0, a pole of order m, then $c_n = \frac{An^{m-1}}{z_0^n}[1+\phi(n)]$ where $\lim_{n\to\infty} \phi(n) = 0$, and A is a constant.

HINT. Expand in a series of powers of z the difference between the function $f(z)$ and the principal part of the expansion of $f(z)$ in the neighbourhood of the pole z_0.

646. Prove that if the radius of convergence of the series $\sum_{n=0}^{\infty} c_n z^n = f(z)$ is equal to unity, and all the singular points of the function $f(z)$ on the circumferenee $|z| = 1$ are poles of orders not exceeding m, then $|c_n| < An^{m-1}$, where A is a constant (if $m = 1$, then $|c_n| < A$, that is, the sequence of coefficients $\{c_n\}$ is bounded).

647. Prove that if the only singular point z_0 on the circumference of the circle of convergence of the power series $f(z) = \sum_{n=0}^{\infty} c_n z^n$ is a pole, then

$$\lim_{n\to\infty} \frac{c_n}{c_{n+1}} = z_0$$

REMARK. Hence, in particular, it follows that the formula $R = \lim_{n\to\infty} \frac{c_n}{c_{n+1}}$ for the radius of convergence always holds if the only singular point on the circle of convergence is a pole.

648. Prove that the assertion of the preceding problem remains true if all the singular points of $f(z)$ on the circumference of the circle of convergence are poles, the order of one of them ($z = z_0$) being higher than that of all the others.

§ 3. Integral functions†

An analytic function is said to be an *integral function* if its only singularity is the point at infinity. If the function $f(z)$ is an integral function and the point $z = \infty$ is a pole, then $f(z)$ is a polynomial; if, however, the point $z = \infty$ is an essential singularity then $f(z)$ is called a *transcendental integral function.*

Let $M(r) = \max_{|z|=r} |f(z)|$. The number $\varrho = \overline{\lim_{r\to\infty}} \log\log M(r)/\log r$ is called the *order* of the integral function $f(z)$. If $\varrho < \infty$, the function $f(z)$ is said to be a function of *finite order*; if $\varrho = \infty$, $f(z)$ is a function of *infinite order.* If $\varrho > 0$, the number $\sigma = \overline{\lim_{r\to\infty}} \dfrac{\log M(r)}{r^\varrho}$ is known as the *type* of the function. If $\sigma = 0$, the function $f(z)$ is said to be a function of *minimal type*; if $\sigma = \infty$, it is a function of *maximal type*; if $0 < \sigma < \infty$ it is a function of *normal type.*

649. Prove the following assertions:

(1) If $\varrho \neq \infty$ and $\sigma \neq \infty$ are respectively the order and type of the function $f(z)$, then for any $\varepsilon > 0$ it is possible to find a number $R(\varepsilon)$ such that the inequalities

$$M(r) < e^{r^{\varrho+\varepsilon}}, \qquad M(r) < e^{(\sigma+\varepsilon)r^\varrho},$$

hold for $r > R$.

It is also possible to find sequences of numbers $\{r_n\}$ and $\{r_n'\}$ converging to infinity for which

$$M(r_n) > e^{r_n'^{\varrho-\varepsilon}} \quad \text{and} \quad M(r_n') > e^{(\sigma-\varepsilon)r_n'^{\varrho}}.$$

(2) If for some natural number k

$$0 < \lim_{r\to\infty} \frac{M(r)}{r^k} < \infty,$$

then $f(z)$ is a polynomial of degree k.

HINT. Use Cauchy's inequalities for the coefficients of the power series $f(z) = \sum_{n=0}^{\infty} c_n z^n$.

† On the subject of this group of problems see, for example, [1, Chapter VII § 1] or B. YA. LEVIN, *The distribution of the roots of integral functions* (*Raspredeleniye kornei tselykh funktsii*), Gostekhizdat, 1956.

(3) if $f(z)$ is a transcendental integral function, then

$$\lim_{r\to\infty}\frac{\log M(r)}{\log r}=\infty.$$

In problems **650-662** determine the order and type of the given functions (n is a natural number).

650. $c_0z_n+c_1z^{n-1}+\ldots+c_n$.

651. e^{az^n} $(a>0)$.

652. z^ne^{3z}.

653. $z^2e^{2z}-e^{3z}$.

654. $e^{5z}-3e^{2z^3}$.

655. $e^{(2-i)z^2}$.

656. $\sin z$.

657. $\cosh z$.

658. $e^z\cos z$.

659. $\cos\sqrt{z}$.

660*. $\displaystyle\sum_{n=0}^{\infty}\frac{z^n}{(2^m\cdot n)!}$ (m a natural number).

661. e^{e^z}.

662*. $\displaystyle\int_0^1 e^{zt^3}dt$.

663. The integral function $f(z)$ is of order ϱ and type σ $(0\leqslant\sigma\leqslant\infty)$. Prove that the function $P(z)f(z)+Q(z)$, where $P(z)$ and $Q(z)$ are any polynomials is also of order ϱ and type σ.

664. The integral functions $f_1(z)$ and $f_2(z)$ are of orders ϱ_1 and ϱ_2 respectively, where $\varrho_1\neq\varrho_2$. What can be said of the order ϱ^* of the functions $f_1(z)f_2(z)$ and $f_1(z)+f_2(z)$?

665. The integral functions $f_1(z)$ and $f_2(z)$ are of the same order ϱ and are respectively of types equal to σ_1 and σ_2, where $\sigma_1\neq\sigma_2$. What can be said of the order ϱ^* and type σ^* of the functions:

(1) $f_1(z)f_2(z)$; (2) $f_1(z)+f_2(z)$?

666. The integral functions $f_1(z)$ and $f_2(z)$ are of the same order ϱ and of the same type σ. What can be said of the order ϱ^* and type σ^* of the functions:

(1) $f_1(z)f_2(z)$; (2) $f_1(z)+f_2(z)$?

667*. Prove that the order and type of an integral function are not changed on differentiating the function.

Solve problems **668-675** by the use of the following theorem.

If the Taylor expansion of an integral function is of the form

$f(z) = \sum_{n=0}^{\infty} c_n z^n$, then the order and type of this function are given by the formulae:

$$\varrho = \overline{\lim_{n\to\infty}} \frac{n \log n}{\log \left| \frac{1}{c_n} \right|},$$

$$(\sigma e \varrho)^{\frac{1}{\varrho}} = \overline{\lim_{n\to\infty}} \left(n^{\frac{1}{\varrho}} \sqrt[n]{|C_n|} \right).$$

668. Prove that the integral function

$$f(z) = \sum_{n=0}^{\infty} \frac{(Az)^n}{\Gamma(\alpha n+1)} \qquad (A > 0,\ \alpha > 0)$$

is of order $\varrho = \frac{1}{\alpha}$ and type $\sigma = A^{\frac{1}{\alpha}}$.

HINT. Use Stirling's formula

$$\Gamma(\alpha n+1) = \left(\frac{\alpha n}{e}\right)^{\alpha n} \sqrt{(2\pi\alpha n)} \left[1 + O\left(\frac{1}{n}\right)\right].$$

In problems **669–675** find the orders and types of the given functions.

669. $f(z) = \sum_{n=1}^{\infty} \left(\frac{z}{n}\right)^n$.

670. $f(z) = \sum_{n=1}^{\infty} \left(\frac{\log n}{n}\right)^{\frac{n}{a}} z^n \qquad (a > 0)$.

671. $f(z) = \sum_{n=2}^{\infty} \left(\frac{1}{n \log n}\right)^{\frac{n}{a}} z^n \qquad (a > 0)$.

672. $f(z) = \sum_{n=0}^{\infty} e^{-n^2} z^n$.

673. $f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^{n^{1+n}}} \qquad (a > 0)$.

674. $f(z) = \sum_{n=0}^{\infty} \frac{\cosh \sqrt{n}}{n!} z^n$.

675. $z^{-\nu} J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{n! \Gamma(n+\nu+1)}$ $(\nu > -1$; $J_\nu(z)$ is a Bessel function of the νth order).

If ϱ is the order of the integral function $f(z)$, then the function $h(\phi) = \overline{\lim_{r\to\infty}} \frac{\log |f(re^{i\phi})|}{r^\varrho}$ is called the *indicator function* of the function $f(z)$.

In problems **676-681** find the indicator functions of the given functions.

676. e^z.

677. $\sin z$.

678. $\cos z$.

679. $\cosh z$.

680. e^{z^n}.

681. $e^z + z^2$.

682. The integral function $f(z)$ has the indicator function $h(\phi)$. What is the indicator function $h^*(\phi)$ of the function $f(z) + P(z)$, where $P(z)$ is a polynomial?

CHAPTER VI

VARIOUS SERIES OF FUNCTIONS. PARAMETRIC INTEGRALS. INFINITE PRODUCTS

§ 1. Series of functions

In problems **683-692** find the domains of convergence of the given series.

683. $\sum_{n=0}^{\infty}\left(z^n+\frac{1}{2^n z^n}\right)$.

684. $\sum_{n=0}^{\infty}\left(\frac{z^n}{n!}+\frac{n^2}{z^n}\right)$.

685. $\sum_{n=1}^{\infty}\left[\frac{z(z+n)}{n}\right]^n$.

686. $\sum_{n=1}^{\infty} e^{z\log n}$.

687. $\sum_{n=1}^{\infty}\frac{\sin nz}{n}$.

688. $\sum_{n=0}^{\infty}\frac{(-1)^n}{z+n}$.

689. $\sum_{n=0}^{\infty}\frac{2^n}{z^{2^n}+1}$.

690. $\sum_{n=1}^{\infty}\frac{z^n}{1-z^n}$.

691. $\sum_{n=0}^{\infty}\frac{z^n}{1+z^{2n}}$.

692. $\sum_{n=1}^{\infty}\frac{z^n}{(4+z)(4+z^2)\ldots(4+z^n)}$.

693*. Prove that if the series $\sum_{n=1}^{\infty} a_n$ converges then the series $\sum_{n=1}^{\infty}\frac{a_n z^n}{1-z^n}$ converges everywhere where $|z|\neq 1$; if, however, the series

$\sum_{n=1}^{\infty} a_n$ diverges then the series $\sum_{n=1}^{\infty} \frac{a_n z^n}{1-z^n}$ converges in the circle of convergence of the series $\sum_{n=1}^{\infty} a_n z^n$ and diverges outside this circle.

694. (1) Expand in a sum of powers of z the sum of the series $\sum_{n=1}^{\infty} \frac{a_n z^n}{1-z^n}$; find the radius of convergence of the series obtained.

(2) Prove that for $|z| < 1$

$$\sum_{n=1}^{\infty} \phi(n) \frac{z^n}{1-z^n} = \frac{z}{(1-z)^2},$$

where $\phi(n)$ is the number of those positive integers less than and coprime to n.

HINT. Use the relation $\sum \phi(n) = m$ where n assumes in succession the value of each of the divisors of the number m, including 1 and m.

695. Expand the function $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} = \sum_{n=1}^{\infty} e^{-z \log n}$ (Riemann's Zeta function) in a Taylor series in the neighbourhood of the point $z = 2$ and find its radius of convergence.

In problems **696–699** find the sum of the given series.

696. $\sum_{n=1}^{\infty} \left(\frac{1}{1+z^n} - \frac{1}{1+z^{n-1}} \right)$ $(|z| \neq 1)$.

697. $\sum_{n=1}^{\infty} \frac{z^n}{(1-z^n)(1-z^{n+1})}$ $(|z| \neq 1)$.

HINT. Multiply numerator and denominator by $(1 - z)$.

698. $\sum_{n=1}^{\infty} \frac{z^{2^n}}{\prod_{k=0}^{n} (1+z^{2^k})}$.

699. $\sum_{n=1}^{\infty} \frac{z^{2^n-1}}{z^{2^n}-1}$.

The series $\sum_{k=1}^{\infty} f_k(z)$ is said to be *uniformly convergent* on the set E if for any $\varepsilon > 0$ it is possible to find a number $N(\varepsilon)$ such that the inequality $\left|\sum_{k=n}^{\infty} f_k(z)\right| < \varepsilon$ is satisfied for all $n > N(\varepsilon)$ and for all points z of the set E.

700. Prove the propositions:

(1) For the uniform convergence of the series $\sum_{n=1}^{\infty} f_n(z)$ on the set E it is necessary and sufficient that for any $\varepsilon > 0$ a number $N = N(\varepsilon)$ should exist such that for all $n > N$, all $z \in E$ and any natural number p the inequality

$$\left|\sum_{k=n+1}^{n+p} f_k(z)\right| < \varepsilon$$

should be satisfied.

(2) From the uniform convergence of the series $\sum_{n=1}^{\infty} |f_n(z)|$ on the set E there follows the uniform convergence on the same set of the series $\sum_{n=1}^{\infty} f_n(z)$.

701. Find the sets on which the given sequences converge uniformly:

$$(1)\ \left\{\frac{1}{1+z^n}\right\};\qquad (2)\ \left\{\frac{z^n}{1+z^{2n}}\right\};\qquad (3)\ \left\{\frac{\sin nz}{n}\right\}.$$

702. Prove:

In order that the sequence of continuous functions $\{f_n(z)\}$ should converge uniformly on the bounded closed set E, it is necessary and sufficient that this sequence should converge at all the points of this set and that it should converge continuously at all the limit points of the set E, that is, that for every sequence of points z_n, belonging to the set E and converging to the point z_0,

$$\lim_{n\to\infty} f_n(z_n) = f(z_0).$$

In problems **703–707** find the sets on which the given series converge uniformly.

703. $\sum_{n=1}^{\infty} \frac{1}{n^2}\left(z^n+\frac{1}{z^n}\right)$.

704. $\sum_{n=0}^{\infty} e^{-nz}$.

705. $\sum_{n=1}^{\infty} e^{-z\log n}$.

706. $\sum_{n=1}^{\infty} (\sin nz)/n^2$.

707. $\sum_{n=1}^{\infty} (\sin nz)/n$.

708. Prove that the series $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$ converges uniformly in the closed circle $|z| \leqslant 1$. Does the series obtained by term-by-term differentiation converge uniformly in the circle $|z| < 1$?

709. Prove that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{z+n}$ converges uniformly in any finite part of the plane, from which there has been removed disks of arbitrarily small radius ϱ with centres at the points $z = 0$, -1, -2, ...

Prove that this series does not converge absolutely at a single point.

710. Prove that the series $\sum_{n=1}^{\infty} \frac{z^n}{n}$ converges uniformly in the interval $(-1, 0)$, and the series $\sum_{n=1}^{\infty} \left|\frac{z^n}{n}\right|$ converges in the same interval, but not uniformly. (Hence the series $\sum_{n=1}^{\infty} \frac{z^n}{n}$ cannot be majorised in the interval $(-1, 0)$ by a convergent series of numbers.)

REMARK. This example shows that Weierstrass's sufficient test for uniformity of convergence is not necessary.

711. (1) The series $\sum_{n=0}^{\infty} \frac{z}{(1+z^2)^n}$ converges absolutely for $|z| \geqslant 0$, $|\arg z| \leqslant \pi/4$ (these values of z do not exhaust the whole of the domain of absolute eonvergence which, as is easily seen, consists of the point $z = 0$ and the outside of the lemniscate $|1+z^2| = 1$). Prove that the series converges non-uniformly in the given domain.

REMARK. This shows that uniform convergence does not follow even from the absolute convergence of a series in a closed region.

(2) Prove that in the same region the series $\sum_{n=0}^{\infty} \frac{(-1)^n z}{(1+z^2)^n}$ converges uniformly and absolutely, but not absolutely uniformly (that is, the series of moduli does not converge uniformly).

712. Prove that if the series $\sum_{n=1}^{\infty} |f_n(z)|$ converges uniformly in every closed region inside the domain G, then the series $\sum_{n=1}^{\infty} |f_n'(z)|$ also possesses the same property.

§ 2. Dirichlet series†

Series of the form $\sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$, where the a_n are complex coefficients and the λ_n are real non-negative numbers, satisfying the conditions

$$\lambda_1 < \lambda_2 < \ldots \text{ and } \lim_{n\to\infty} \lambda_n = \infty,$$

are known as *Dirichlet series.*

713. Prove that if a Dirichlet series converges at the point $z_0 = x_0 + iy_0$, then it converges at all the points of the half-plane $\operatorname{Re} z > \operatorname{Re} z_0$, the convergence being uniform in every angle $|\arg(z-z_0)| \leqslant \theta < \pi/2$.

HINT. Apply Abel's transformation to the sum

$$\sum_{n=p}^{q} a_n e^{-\lambda_n z} = \sum_{n=p}^{q} a_n e^{-\lambda_n z_0} e^{-\lambda_n (z-z_0)}$$

and use the inequality ($a < b$, $z = x + iy$)

$$|e^{-az} - e^{-bz}| = \left| z \int_a^b e^{-zt}\, dt \right| \leqslant \frac{|z|}{x} (e^{-ax} - e^{-bx}).$$

714. Prove that if a Dirichlet series converges absolutely at the point $z = z_0$, then it converges absolutely and uniformly in the half-plane $\operatorname{Re} z \geqslant \operatorname{Re} z_0$.

From the theorems formulated in problems **713–714** it follows that the domain of convergence of a Dirichlet series (if it exists) is a half-plane $\operatorname{Re} z > x_c$ ($x_c \geqslant -\infty$), and the domain of absolute convergence (if it exists)

† On the subject of this group of problems see, for example, [1, Chapter IV, § 1].

is a half-plane $\operatorname{Re} z > x_a$ ($x_a \geqslant -\infty$), the series either converging absolutely on the whole of the straight line $\operatorname{Re} z = x_a$, or not converging absolutely at a single point of this straight line. The numbers x_c and x_a are called respectively the *abscissa of convergence* and the *abscissa of absolute convergence* of the Dirichlet series.

In problems **715–721** find the abscissas of convergence (x_c) and the abscissas of absolute convergence (x_a) of the given series.

715. $\sum_{n=0}^{\infty} e^{-n^3} e^{-zn^3}$

716. $\sum_{n=2}^{\infty} \frac{(-1)^n}{n} e^{-z \log \log n}$.

717. $\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}} e^{-z \log \log n}$.

718. $\sum_{n=2}^{\infty} (-1)^n e^{-z \log \log n}$.

719. $\sum_{n=1}^{\infty} (-1)^n e^{-z \log n}$.

720. $\sum_{n=1}^{\infty} \frac{1}{n^2} e^{-z \log n}$.

721. $\sum_{n=0}^{\infty} e^{e^n} e^{-zn^2}$.

722. Prove that if $\lim_{n\to\infty} (\log n)/\lambda_n = 0$, then

$$x_c = x_a = \overline{\lim_{n\to\infty}} \frac{\log |a_n|}{\lambda_n}.$$

723. Prove that if $\overline{\lim_{n\to\infty}} (\log n)/\lambda_n = l$, then $x_a - x_c \leqslant l$.

In problems **724–728** investigate the convergence of the Dirichlet series on the boundary of the half-plane of convergence.

724. $\sum_{n=1}^{\infty} (-1)^n e^{-z \log n}$.

725. $\sum_{n=1}^{\infty} \frac{1}{n^3} e^{-z \log n}$.

726. $\sum_{n=1}^{\infty} \frac{1}{n} e^{-zn}$.

727. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} e^{-zn}$.

728. $\sum_{n=1}^{\infty} \frac{(-1)^{[\sqrt{n}]}}{n} e^{-zn}$.

HINT. See problem **479**.

The series $\sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$, where the λ_n are complex numbers ($|\lambda_1| \leqslant |\lambda_2| \leqslant \ldots$; $\lim_{n\to\infty} \lambda_n = \infty$), is known as a *generalised Dirichlet series.*

729. Let the numbers λ_n satisfy the conditions

$$\lim_{n\to\infty} \frac{\log n}{\lambda_n} = 0 \quad \text{and} \quad \overline{\lim_{n\to\infty}} \frac{\log |a_n|}{|\lambda_n|} = k < \infty.$$

Prove that if $\alpha \leqslant \arg \lambda_n \leqslant \beta$, then the generalised Dirichlet series converges absolutely at every point $z = x + iy$ for which the inequality $x \cos \phi - y \sin \phi - k > 0$, is satisfied for all ϕ of $[\alpha, \beta]$, and diverges at a point for which for all ϕ of $[\alpha, \beta]$

$$x \cos \phi - y \sin \phi - k < 0.$$

730. An arbitrary generalised Dirichlet series $\sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$ is given. Let $k(\phi, \alpha) = \overline{\lim_{k\to\infty}} \frac{\log |a_{n_k}|}{|\lambda_{n_k}|}$ and $k(\phi) = \lim_{\alpha\to 0} k(\phi, \alpha)$, where $\{n_k\}$ is the sequence of all the suffixes for which $\phi - \alpha \leqslant \arg \lambda_{n_k} \leqslant \phi + \alpha$ (if no such subsequence $\{n_m\}$ exists for which $\lim_{m\to\infty} \arg \lambda_{n_m} = \phi$, then it is necessary to put $k(\phi) = -\infty$).

Prove that if $\lim_{n\to\infty} (\log n)/\lambda_n = 0$, then the series converges absolutely within the domain G, the points $z = x + iy$ of which satisfy for any ϕ the condition $x \cos \phi - y \sin \phi - k(\phi) > 0$, and diverges at every point lying outside G.

§ 3. Parametric integrals

731. Prove the theorem:

Let C be a simple contour (closed or not), possessing a finite length, and $f(\tau, z)$ a function, analytic with respect to the variable z and continuous with respect to τ for every z of some domain D and for all the points τ, belonging to the contour C. Then the function represented by the integral

$$F(z) = \int_C f(\tau, z) \, d\tau,$$

is an analytic function of the variable z and

$$F'(z) = \int_C f'_z(\tau, z) \, d\tau.$$

If the integral $\int_C f(\tau, z)\mathrm{d}\tau$ is improper, that is, if the integrand has discontinuities at some isolated values $\tau \in C$ or the contour of integration includes the point at infinity, then the definitions of convergence and uniform convergence of such an integral are exactly similar to the corresponding definitions given in books on mathematical analysis.

732. Prove that for the uniform convergence of the integral $\int_C f(\tau, z)\mathrm{d}\tau$ on the set E with respect to some point $\tau_0 \neq \infty$ of the contour C it is necessary and sufficient that for any $\varepsilon > 0$ there should exist a number $\delta(\varepsilon)$ such that

$$\left| \int_{C_\sigma} f(\tau, z)\mathrm{d}\tau \right| < \varepsilon$$

for all the points z of the set E and for every arc C_σ of the contour C, lying in a δ-neighbourhood of the point τ_0 and not containing this point either as an interior or as an end point.

733. Formulate and prove a similar criterion for uniform convergence of the integral if $\tau_0 = \infty$. Consider the cases when the contour C is not bounded at one end, or at both ends.

734. Prove that if $|f(\tau, z)| \leqslant |\psi(\tau)|$ for all the points z of the set E and if $\int_C |\psi(\tau)|\mathrm{d}\tau$ converges then the integral $\int_C f(\tau, z)\mathrm{d}\tau$ converges uniformly on the set E.

735. Let $f(\tau, z)$ be a function analytic with respect to z and continuous with respect to τ for all the points z, belonging to some domain D, and points τ, belonging to the contour C, with the exception of some isolated points of it where the conditions imposed on the function $f(\tau, z)$ are not satisfied either for all points z, or only for some of them.

Prove that if the improper integral

$$F(z) = \int_C f(\tau, z)\mathrm{d}\tau$$

converges locally uniformly in the domain D (that is, in every closed sub-region of the domain D), the function $F(z)$ is analytic and

$$F'(z) = \int_C \frac{\partial f}{\partial z}\mathrm{d}\tau,$$

the latter integral converging uniformly within D.

In problems **736–743** find the sets on which the given integrals converge uniformly.

736. $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\,dt \qquad (t^{z-1} = e^{(z-1)\log t})$.

737. $\int_0^\infty e^{-zt^2}\,dt$.

738. $\int_0^\infty \frac{\sin t}{t^z}\,dt$.

739. $\int_0^\infty \frac{\cos t}{t^z}\,dt$.

740. $\int_0^\infty \frac{\sin tz}{t}\,dt$.

741. $\int_{c-i\infty}^{c+i\infty} \frac{e^{zt}}{t}\,dt \qquad (c \neq 0)$.

742. $\int_c^{c+i\infty} \frac{e^{zt}}{t}\,dt \qquad (c \neq 0)$.

743. $\int_{c-i\infty}^{c+i\infty} \frac{z^t}{t}\,dt \qquad (c \neq 0, \quad z^t = e^{t\log z})$.

An integral of the form

$$\int_0^\infty e^{-tz} f(t)\,dt, \tag{1}$$

where the function $f(t)$ is integrable on the segment $[0, a]$ for any positive $a < \infty$ is called a *Laplace integral*.

744. Prove the following propositions:

(1) If the integral (1) converges at the point $z = z_0$ then it converges in the half-plane $\operatorname{Re} z > \operatorname{Re} z_0$, the convergence being uniform in the angle $|\arg(z - z_0)| \leqslant \theta < \pi/2$.

(2) If the integral (1) converges absolutely for $z = z_0$ then it converges absolutely and uniformly in the half-plane $\operatorname{Re} z \geqslant \operatorname{Re} z_0$.

(3) If $\overline{\lim_{t\to\infty}} \frac{\log|f(t)|}{t} = \beta$, then the integral (1) converges absolutely in the half-plane $\operatorname{Re} z > \beta$ and uniformly in every half-plane $\operatorname{Re} z \geqslant \beta + \varepsilon$ $(\varepsilon > 0)$ (construct an example of a Laplace integral which converges absolutely in the whole plane and for which $\beta = \infty$).

(4) If $\lim_{t\to\infty} \frac{\log|f(t)|}{t} = \alpha$, then the integral (1) does not converge absolutely at a single point of the half-plane $\operatorname{Re} z < \alpha$.

It follows from the theorem formulated in problem **744** that the domains of convergence and of absolute convergence of a Laplace integral (if they exist) are the half-planes $\operatorname{Re} z > x_c$ and $\operatorname{Re} z > x_a$; the number x_c is called the *abscissa of convergence*, and x_a is called the *abscissa of absolute convergence* of the Laplace integral.

In problems **745–751** find x_c and x_a for the integral $\int_0^\infty e^{-zt} f(t)\,dt$, where $f(t)$ is the given function.

745. $f(t) = 1$. **746.** $f(t) = e^{-t^2}$. **747.** $f(t) = e^{t^2}$.

748. $f(t) = e^{-t^2}$ for $0 \leqslant t < \log\log 3$ and $\log\log 2k \leqslant t < \log\log(2k+1)$ $(k = 2, 3, \ldots)$ and $f(t) = -e^{-t^2}$ for $\log\log(2k+1) \leqslant t < \log\log(2k+2)$ $(k = 1, 2, \ldots)$.

749. $f(t) = e^{\frac{1}{2}e^t}$ for $0 \leqslant t < \log\log 3$ and $\log\log 2k \leqslant t < \log\log(2k+1)$ $(k = 2, 3, \ldots)$ and $f(t) = -e^{\frac{1}{2}e^t}$ for $\log\log(2k+1) \leqslant t < \log\log(2k+2)$ $(k = 1, 2, \ldots)$.

750. $f(t) = e^{e^t}$ for $0 \leqslant t < \log\log 3$ and $\log\log 2k \leqslant t < \log\log(2k+1)$ $(k = 2, 3, \ldots)$ and $f(t) = -e^{e^t}$ for $\log\log(2k+1) \leqslant t < \log\log(2k+2)$ $(k = 1, 2, \ldots)$.

751. $f(t) = e^t$ for $\log(2k-1) \leqslant t < \log 2k$ $(k = 1, 2, \ldots)$ and $f(t) = -e^t$ for $\log 2k \leqslant t < \log(2k+1)$ $(k = 1, 2, \ldots)$.

In problems **752–755** investigate the convergence of the Laplace integrals $\int_0^\infty e^{-zt} f(t)\,dt$ on the boundary of the half-plane of convergence.

752. $f(t) = 1$.

753. $f(t) = 0$ for $0 \leqslant t \leqslant 1$, $f(t) = 1/t^2$ for $t > 1$.

754. $f(t) = 0$ for $0 \leqslant t \leqslant 1$, $f(t) = 1/t$ for $t > 1$.

755. $f(t) = 0$ for $t = 0$, $f(t) = 1$ for $0 < t \leqslant 1$, and for $t > 1$ $f(t)$ is defined as follows:

$$f(t+1) = f(t)+1, \text{ if } (2k-1)^2 < t+1 \leqslant (2k)^2$$

and

$$f(t+1) = f(t)-1, \text{ if } (2k)^2 < t+1 \leqslant (2k+1)^2 \quad (k = 1, 2, \ldots).$$

§ 4. Infinite products

In problems **756–762** prove the given equalities.

756. $\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right) = \frac{1}{2}$.

757. $\prod_{n=1}^{\infty}\left(1+\frac{1}{n(n+2)}\right)=2$.

758. $\prod_{n=3}^{\infty}\frac{n^2-4}{n^2-1}=\frac{1}{4}$.

759. $\prod_{n=2}^{\infty}\left(1-\frac{2}{n(n+1)}\right)=\frac{1}{3}$.

760. $\prod_{n=2}^{\infty}\frac{n^3-1}{n^3+1}=\frac{2}{3}$.

761. $\prod_{n=1}^{\infty}\left[1+\frac{(-1)^{n+1}}{n}\right]=1$.

762. $\prod_{n=1}^{\infty}\frac{e^{\frac{1}{n}}}{1+\frac{1}{n}}=e^C$ where $C=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\log n\right)$ is Euler's constant.

763. Prove that $\prod_{n=1}^{\infty}\cos\frac{\alpha}{2^n}=\frac{\sin\alpha}{\alpha}$.

HINT. First prove the identity

$$\sin\phi=2^k\sin\frac{\phi}{2^k}\prod_{n=1}^{k}\cos\frac{\phi}{2^n}.$$

764. Using the solution of problem **763** prove that

$$\frac{2}{\sqrt{2}}\,\frac{2}{\sqrt{(2+\sqrt{2}}}\,\frac{2}{\sqrt{[2+\sqrt{(2+\sqrt{2})}]}}\cdots=\frac{\pi}{2}.$$

765.* Prove Wallis's formula

$$\frac{\pi}{2}=\prod_{n=1}^{\infty}\left(\frac{2n}{2n-1}\,\frac{2n}{2n+1}\right).$$

766. Prove that if, as usual, we take $-\pi<\arg p_n\leqslant\pi$, then the infinite product $\prod_{n=1}^{\infty}p_n$ converges and diverges together with the series $\sum_{n=1}^{\infty}\log p_n$.

767. Examine whether the assertion of the preceding problem is still true if it is assumed that:

(1) $0\leqslant\arg p_n<2\pi$; (2) $\alpha<\arg p_n\leqslant\alpha+2\pi$ $(\alpha<0)$.

768. Prove that for the absolute convergence of the infinite product $\prod_{n=1}^{\infty}(1+a_n)$ (that is, for the absolute convergence of the series $\sum_{n=1}^{\infty}\log(1+a_n)$) it is necessary and sufficient that the series $\sum_{n=1}^{\infty}a_n$ should converge absolutely.

769. The infinite products $\prod_{n=1}^{\infty}p_n$ and $\prod_{n=1}^{\infty}q_n$ converge. Investigate the convergence of the infinite products:

$$(1)\ \prod_{n=1}^{\infty}(p_n+q_n);\quad (2)\ \prod_{n=1}^{\infty}(p_n-q_n);\quad (3)\ \prod_{n=1}^{\infty}p_nq_n;\quad (4)\ \prod_{n=1}^{\infty}\frac{p_n}{q_n}.$$

In problems **770–774** investigate the convergence and absolute convergence of the given infinite products.

770. $\prod_{n=1}^{\infty}\left[1+\frac{(-1)^{n+1}}{n}\right]$.

771. $\prod_{n=1}^{\infty}n^{(-1)^n}$.

772. $\prod_{n=1}^{\infty}\left[1+\frac{(-1)^{n+1}}{n^p}\right]$ $(p>0)$.

773. $\prod_{n=1}^{\infty}\left(1+\frac{1}{n^p}\right)$ $(p>0)$.

774. $\prod_{n=1}^{\infty}\cos z_n$, if it is known that the series $\sum_{n=1}^{\infty}|z_n|^2$ converges.

---

775. Prove that inside the unit circle

$$\prod_{n=0}^{\infty}(1+z^{2^n})=\frac{1}{1-z},$$

the infinite product converging absolutely.

In problems **776–784** find the domain of convergence of the infinite products.

776. $\prod_{n=1}^{\infty}(1-z^n)$.

777. $\prod_{n=1}^{\infty}\left(1-\frac{z^n}{2^n}\right)$.

778. $\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)$.

779. $\prod_{n=2}^{\infty}\left[1-\left(1-\frac{1}{n}\right)^{-n}z^{-n}\right]$.

780. $\prod_{n=1}^{\infty}\left[1+\left(1+\frac{1}{n}\right)^{n^2}z^n\right]$.

781. $\prod_{n=1}^{\infty}\cos\frac{z}{n}$.

782. $\prod_{n=1}^{\infty}\frac{\sin\frac{z}{n}}{z/n}$.

783. $\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}$.

784. $\prod_{n=1}^{\infty}(1+c_nz)$, if it is known that the series $\sum_{n=1}^{\infty}|c_n|$ converges.

---

785. Prove that the infinite product

$$\prod_{n=1}^{\infty}\left[1+\frac{(-1)^{n+1}}{n^z}\right] \qquad (n^z=e^{z\log n})$$

converges in the half-plane $\operatorname{Re} z>1/2$ and converges absolutely in the half-plane $\operatorname{Re} z>1$.

786. $\{f_n(z)\}$ is a sequence of functions analytic in the domain G, and all these functions, with the exception of a finite number of them, do not become zero in the domain G. Prove that if $|f_n(z)|\leqslant\alpha_n$ for all $z\in G$, where α_n is independent of z, and the series $\sum_{n=1}^{\infty}\alpha_n$ converges, then the function

$$F(z)=\prod_{n=1}^{\infty}[1+f_n(z)]$$

is analytic in the domain G.

In problems **787–790** some of the properties of the Gamma function which follow from its definition as the limit of an infinite product are explained (see, for example, [1, Chapter VII, § 4] or [3, Chapter VII, § 1]).

787. Prove that the infinite product

$$\Gamma(z+1)=\prod_{n=1}^{\infty}\frac{n}{z+n}\left(\frac{n+1}{n}\right)^z \qquad \left(\left(\frac{n+1}{n}\right)^z=e^{z\log\frac{n+1}{n}}\right)$$

converges absolutely in the whole plane, except for negative integral values of z, and represents a function analytic in the whole plane except at the points $z = -1, -2, \ldots$

788. Prove Euler's formula

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\, n^z}{z(z+1)(z+2)\ldots(z+n)} \qquad (n^z = e^{z \log n})$$

and show that:

(1) $\Gamma(z+1) = z\Gamma(z)$;

(2) $\Gamma(m+1) = m!$, if m is a natural number.

789. Prove that

$$\prod_{n=1}^{\infty} \frac{n(\alpha+\beta+n)}{(\alpha+n)(\beta+n)} = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} \qquad (\alpha,\beta \neq -1, -2, \ldots).$$

790. Prove Weierstrass's formula

$$\frac{1}{\Gamma(z+1)} = e^{Cz} \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right) e^{-\frac{z}{n}},$$

where C is Euler's constant.

HINT. Use the solution of problem **762**.

791*. Let $p_1, p_2, \ldots, p_n, \ldots$ be the sequence of all the prime numbers ($p_1 = 2$, $p_2 = 3$, $p_3 = 5, \ldots$). $\zeta(s) = \sum_{n=1}^{\infty} n^{-s} (n^{-s} = e^{-s \log n})$ (Riemann's Zeta function) is analytic in the half-plane Re $s > 1$ (see problem **705**). Prove that:

(1) $$\zeta(s) = \frac{1}{\prod_{n=1}^{\infty} (1-p_n^{-s})};$$

(2) The function $\zeta(s)$ has no zeros in the half-plane Re $s > 1$.

REMARK. An extensive literature is devoted to Riemann's Zeta function. See, for example, the monograph: E. C. TITCHMARSH, *The zeta-function*, Cambridge Tracts, No. 26.

792*. Prove that the series $\sum_{n=1}^{\infty} \frac{1}{p_n}$, where $\{p_n\}$ is the sequence of prime numbers, diverges.

CHAPTER VII

RESIDUES AND THEIR APPLICATIONS

§ 1. The calculus of residues

In problems **793–813** it is required to find the residues of the given functions at all their isolated singularities and at infinity (if it is not the limit of singularities).

793. $\dfrac{1}{z^3-z^5}$.

794. $\dfrac{z^2}{(z^2+1)^2}$.

795. $\dfrac{z^{2n}}{(1+z)^n}$ (n is a natural number).

796. $\dfrac{1}{z(1-z^2)}$.

797. $\dfrac{z^2+z-1}{z^2(z-1)}$.

798. $\dfrac{\sin 2z}{(z+1)^3}$.

799. $\dfrac{e^z}{z^2(z^2+9)}$.

800. $\tan z$.

801. $\dfrac{1}{\sin z}$.

802. $\cot^2 z$.

803. $\cot^3 z$.

804. (1) $\cos\dfrac{1}{z-2}$;

(2) $z^3\cos\dfrac{1}{z-2}$.

805. $e^{z+\frac{1}{z}}$

806. $\sin z \sin\dfrac{1}{z}$.

807. $\sin\dfrac{z}{z+1}$.

808. $\cos\dfrac{z^2+4z-1}{z+3}$.

809. $\dfrac{1}{z(1-e^{-hz})}$ ($h\neq 0$).

810. $z^n \sin 1/z$ (n is an integer).

811. $\dfrac{1}{\sin\dfrac{1}{z}}$.

812. $\sqrt{z}/\sin\sqrt{z}$.

813. $(\tan z)/z^n$ (n is a natural number).

In problems **814–821** it is required to find the residues of each of the single-valued branches of the corresponding many-valued functions at the given points.

814. $\frac{\sqrt{z}}{1-z}$ at the point $z=1$.

815. $\frac{1}{1+\sqrt{(2-z)}}$ at the point $z=1$.

816. $\frac{z^a}{1-\sqrt{z}}$ $(z^a = e^{a\operatorname{Log} z})$ at the point $z=1$.

817. $\sqrt{\big((z-a)(z-b)\big)}$ at the point $z=\infty$.

818. (1) $\operatorname{Log}\frac{z-\alpha}{z-\beta}$ at the point $z=\infty$;

(2) $e^z \operatorname{Log}\frac{z-\alpha}{z-\beta}$ at the point $z=\infty$.

819. (1) $\operatorname{Log} z \sin\frac{1}{z-1}$ at the point $z=1$.

(2) $\operatorname{Log} z \cos\frac{1}{z-1}$ at the point $z=1$.

820. $\frac{\tan^{-1} z}{z}$ at the points $z=0$ and $z=\infty$.

821. $z^n \operatorname{Log}\frac{z-\alpha}{z-\beta}$ (n is an integer) at the points $z=0$ and $z=\infty$ (in calculating the residue at the point $z=0$ it is assumed that $\alpha \neq 0$, $\beta \neq 0$).

822. The expansion of a function in the neighbourhood of infinity is of the form

$$f(z) = c_0 + \frac{c_1}{z} + \cdots$$

Find $\operatorname{res}[\{f(z)\}^2]_{z=\infty}$.

823. Find $\operatorname{res}[\phi(z)f(z)]_{z=a}$, if $\phi(z)$ is analytic at the point a, and $f(z)$ has at this point:

(1) A simple pole with residue A;

(2) A pole of order k with principal part $\frac{c_{-1}}{z-a} + \cdots + \frac{c_{-k}}{(z-a)^k}$.

824. Find $\operatorname{res}\left[\dfrac{f'(z)}{f(z)}\right]_{z=a}$ if:

(1) a is a zero of order n of the function $f(z)$;
(2) a is a pole of order n of the function $f(z)$.

825. Find $\operatorname{res}\left[\phi(z)\dfrac{f'(z)}{f(z)}\right]_{z=a}$ if $\phi(z)$ is analytic at the point a and:

(1) a is a zero of order n of the function $f(z)$;
(2) a is a pole of order n of the function $f(z)$.

826. Find $\operatorname{res}\{f[\phi(z)]\}_{z=a}$ if the function $\phi(z)$ is analytic at the point a and $\phi'(a)\neq 0$, and $f(\zeta)$ has a simple pole at the point $\zeta=\phi(a)$ with residue A.

827. At the point a the function $\phi(z)$ has a simple pole with residue A, and $f(\zeta)$ has at infinity a simple pole with principal part $B\zeta$. Find

$$\operatorname{res}\{f[\phi(z)]\}_{z=a}.$$

828. The function $f(z)$, which assumes real values on the arc l of the circle $|z-a|=R$, is continued analytically through this arc by the symmetry principle. Let the point $z=\beta$ $(\beta\neq a)$ be for $f(z)$ a pole of order k with principal part

$$\sum_{n=1}^{k}\frac{c_{-n}}{(z-\beta)^n}.$$

Find $\operatorname{res}[f(z)]_{z=\beta^*}$, where β^* is the point symmetrical to $z=\beta$ with respect to l.

§ 2. The evaluation of integrals

In problems **829–837** evaluate the integrals, assuming that the closed contours are traversed in the positive direction.

829. $\displaystyle\int_C \frac{dz}{z^4+1}$, where C is the circle $x^2+y^2=2x$.

830. $\displaystyle\int_C \frac{z\,dz}{(z-1)(z-2)^2}$, where C is the circle $|z-2|=\frac{1}{2}$.

831. $\displaystyle\int_C \frac{dz}{(z-3)(z^5-1)}$, where C is the circle $|z|=2$.

HINT. Use the fact that the sum of the residues at all the singularities (including the point at infinity) is equal to zero.

832. $\int_C \frac{z^3 dz}{2z^4+1}$, where C is the circle $|z| = 1$.

833. $\int_C \frac{e^z}{z^2(z^2-9)} dz$, where C is the circle $|z| = 1$.

834. $\frac{1}{2\pi i} \int_C \sin \frac{1}{z} dz$, where C is the circle $|z| = r$.

835. $\frac{1}{2\pi i} \int_C \sin^2 \frac{1}{z} dz$, where C is the circle $|z| = r$.

836. $\frac{1}{2\pi i} \int_C z^n e^{\frac{2}{z}} dz$, where n is an integer, and C is the circle $|z| = r$.

837. $\frac{1}{2\pi i} \int_C \frac{f(z) dz}{z g(z)}$, if the simple closed contour C is the boundary of a domain G, containing the point $z = 0$, the function $f(z)$ is analytic in the closed region $\bar{G}$, the function $1/g(z)$ is analytic on C, and in the domain G has no singularities in addition to the simple poles $a_1, a_2, \ldots, a_n$ ($a_k \neq 0$ for any k).

In problems **838–841** evaluate the given integrals.

838. $\frac{1}{2\pi i} \int_C \frac{dz}{\sqrt{(z^2+z+1)}}$, where C is the circle $|z| = r \neq 1$.

839. $\frac{1}{2\pi i} \int_C \frac{dz}{(z^4+1)\sqrt{(z^2+1)}}$ ($\sqrt{1} = 1$), where C is the parabola $y^2 = x$, traversed in the direction of increasing y.

840. $\frac{1}{2\pi i} \int_C \frac{dz}{a^z \sin \pi z}$ ($a^z = e^{z \log a}$), where $a > 0$, and C is the straight line $x = \alpha$, $0 < \alpha < 1$, traversed from below upwards.

HINT. Consider $\frac{1}{2\pi i} \int_\gamma \frac{dz}{a^z \sin \pi z}$, where the contour γ is indicated in Fig. 13, and pass to the limit as $\beta \to \infty$.

841. $\frac{1}{2\pi i} \int_C \frac{e^z dz}{\cos z}$, where the contour of integration C is indicated in Fig. 14.

If the function $f(x)$ becomes infinite at $x = c$ $(a < c < b)$, then the Cauchy principal value of the integral $\int_a^b f(x)\,dx$ is defined by

$$\lim_{\varepsilon \to 0} \left[\int_a^{c-\varepsilon} f(x)\,dx + \int_{c+\varepsilon}^b f(x)\,dx \right].$$

This definition is generalised to the case of a curvilinear integral in an obvious manner.

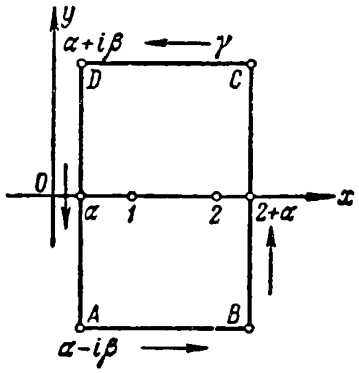

FIG. 13

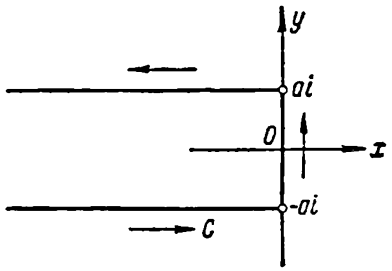

FIG. 14

If the function $f(x)$ is continuous on the whole of the real axis, the principal value of the integral $\int_{-\infty}^{\infty} f(x)\,dx$ is defined as $\lim_{N \to \infty} \int_{-N}^{N} f(x)\,dx$.

In problems **842–849** find the definite integrals. If an integral is improper and diverges, find its principal value (if it exists).

842. $\int_0^{2\pi} \frac{d\phi}{a+\cos\phi}$ $(a > 1)$.

HINT. Put $e^{i\phi} = z$.

843. $\int_0^{2\pi} \frac{d\phi}{(a+b\cos\phi)^2}$ $(a > b > 0)$.

844. $\int_0^{2\pi} \frac{d\phi}{(a+b\cos^2\phi)^2}$ $(a > 0,\ b > 0)$.

845. $\int_0^{2\pi} \frac{d\phi}{1-2a\cos\phi+a^2}$ (a is a complex number and $a \neq \pm 1$).

846. $\int_0^{2\pi} \frac{\cos^2 3\phi \, d\phi}{1-2a\cos\phi+a^2}$ (a is a complex number and $a \neq \pm 1$).

847. $\int_0^{2\pi} e^{\cos\phi} \cos(n\phi - \sin\phi) d\phi$ (n is an integer).

848. $\int_0^{\pi} \tan(x+ia)\,dx$ (a is a real number).

849. $\int_0^{2\pi} \cot(x+a)\,dx$ (a is a complex number and $\operatorname{Im} a \neq 0$).

850. Prove that if $b > a > -1$

$$\int_0^{\frac{\pi}{2}} \cos^a\phi \cos b\phi \, d\phi = \frac{\pi\Gamma(a+1)}{2^{a+1}\Gamma\left(\frac{a+b}{2}+1\right)\Gamma\left(\frac{a-b}{2}+1\right)}$$

HINT. Consider $\int_C \left(z+\frac{1}{z}\right)^a z^{b-1} dz$,

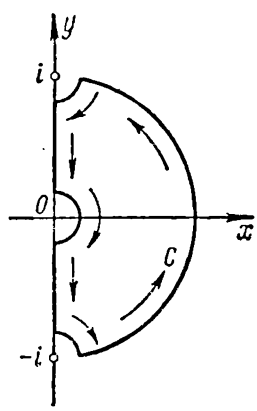

FIG. 15

where C is the contour represented in Fig. 15, and let the radii of the arcs of the small circles tend to zero. In calculating the integral along the vertical segment divide it into two and by corresponding substitutions reduce them

to Eulerian integrals of the first kind; also use the known relation $B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$ and the formula $\Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin \pi p}$.

In problems **851–855** evaluate the integrals with infinite limits.

851. $\int_{-\infty}^{\infty} \frac{x\,dx}{(x^2+4x+13)^2}$.

852. $\int_{0}^{\infty} \frac{x^2\,dx}{(x^2+a^2)^2}$ $(a > 0)$.

853. $\int_{0}^{\infty} \frac{dx}{(x^2+1)^n}$ (n is a natural number).

854. $\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}$ $(a > 0, b > 0)$.

855. $\int_{0}^{\infty} \frac{x^2+1}{x^4+1} dx$.

---

856. Prove that

$$\frac{1}{2\pi i} \int_C \frac{d\tau}{\tau^n \bar{\tau}^k} = \frac{i^{k-n-1}(n+k-2)!}{(2h)^{n+k-1}(k-1)!\,(n-1)!}$$ (n and k are natural numbers),

where C is a straight line parallel to the real axis which cuts off on the imaginary axis a segment equal to h $(h > 0)$.

857. Evaluate the integral

$$\frac{1}{2\pi i} \int_C \frac{d\tau}{\bar{\tau}^k(\tau-z)}$$ (k is a natural number),

where C is the contour of the preceding problem.

In problems **858–861**, using Jordan's lemma (see problem **411**), evaluate the given integrals.

858. (1) $\int_{-\infty}^{\infty} \frac{x \cos x\,dx}{x^2-2x+10}$; (2) $\int_{\infty}^{\infty} \frac{x \sin x\,dx}{x^2-2x+10}$.

859. $\int_{-\infty}^{\infty} \frac{x \sin x \, dx}{x^2+4x+20}$.

860. $\int_{-\infty}^{\infty} \frac{\cos ax}{x^2+b^2} dx$ (a and b are real numbers).

861. $\int_{-\infty}^{\infty} \frac{x \sin ax}{x^2+b^2} dx$ (a and b are real numbers).

---

862. Let $f(z) = e^{imz}F(z)$, where $m > 0$ and the function $F(z)$ possesses the following properties:

(1) In the upper half-plane it has a finite number of singularities $a_1, a_2, \ldots, a_n$;

(2) It is analytic at all the points of the real axis, except the points $x_1, x_2, \ldots, x_m$, which are simple poles;

(3) $F(z) \to 0$ if $z \to \infty$ and $\operatorname{Im} z \geqslant 0$.

Prove that

$$\int_{-\infty}^{\infty} f(x) \, dx = 2\pi i \left\{ \sum_{k=1}^{n} \operatorname{res}[f(z)]_{z=a_k} + \frac{1}{2} \sum_{k=1}^{m} \operatorname{res}[f(z)]_{z=x_k} \right\},$$

where the integrals are understood in the sense of the principal value (with respect to all the points x_k and ∞).

In problems **863-866** find the principal values of the given integrals.

863. $\int_{-\infty}^{\infty} \frac{e^{itx}}{x} dx$ (t is a real number).

864. $\int_{-\infty}^{\infty} \frac{x \cos x \, dx}{x^2-5x+6}$.

865. $\int_{-\infty}^{\infty} \frac{\sin x \, dx}{(x^2+4)(x-1)}$.

866. $\int_{-\infty}^{\infty} \frac{\cos ax}{1+x^3} dx$ ($a \geqslant 0$).

In problems **867–872** evaluate the given integrals (a and b are real numbers).

867. $\int_{0}^{\infty} \frac{x^2-b^2}{x^2+b^2} \frac{\sin ax}{x} dx$.

868. $\int_{0}^{\infty} \frac{\sin ax \, dx}{x(x^2+b^2)}$.

869. $\int_0^\infty \frac{\sin ax\,dx}{x(x^2+b^2)^2}$.

870. $\int_0^\infty \frac{\cos 2ax-\cos 2bx}{x^2}dx$.

871. $\int_0^\infty \frac{\sin^2 x}{x^2}dx$.

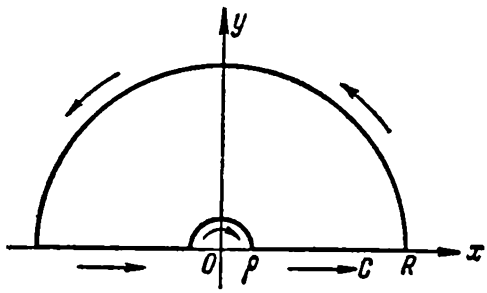

FIG. 16.

HINT. Use the integral $\int_C \frac{e^{2iz}-1}{z^2}dz$, where the contour C is indicated in Fig. 16.

872. $\int_0^\infty \frac{\sin^3 x}{x^3}dx$.

HINT. Use the integral $\int_C \frac{e^{3iz}-3e^{iz}+2}{z^3}dz$, where the contour C is indicated in Fig. 16.

In problems **873–876** evaluate the integrals, considering that $x^p>0$ for $x>0$ (this condition is retained in all the following problems).

873. (1) $\int_0^\infty x^{p-1}\cos ax\,dx$ $(a>0,\ 0<p<1)$;

(2) $\int_0^\infty x^{p-1}\sin ax\,dx$ $(a>0,\ -1<p<1)$.

HINT. Use the integral $\int_C z^{p-1}e^{-az}dz$, where the contour C is indicated in Fig. 17.

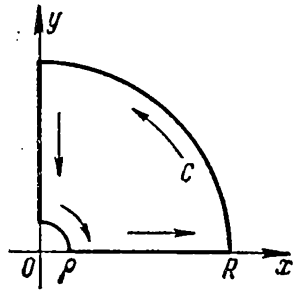

FIG. 17.

874. $\int_0^\infty \cos x^p\,dx \quad (p > 1).$ **876.** $\int_0^\infty \frac{\sin x^p}{x^p}\,dx \quad \left(p > \frac{1}{2}\right).$

875. $\int_0^\infty \sin x^p\,dx \quad (|p| > 1).$

---

877. Let the rational function $f(z)$ have no poles on the positive part of the real axis and

$$\lim_{z\to 0}[z^p f(z)] = \lim_{z\to\infty}[z^p f(z)] = 0.$$

Prove that

$$\int_0^\infty x^{p-1} f(x)\,dx = \frac{\pi}{\sin p\pi}\sum_{k=1}^{n} \operatorname{res}\,[(-z)^{p-1} f(z)]_{z=\alpha_k},$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the poles of the function $f(z)$, p is an integer.

HINT. Consider the integral

$$\int_C (-z)^{p-1} f(z)\,dz \qquad \left((-z)^{p-1} = e^{(p-1)\log(-z)}\right),$$

where C is the contour represented in Fig. 18.

878. Evaluate $\int_0^\infty \frac{dx}{x^p(x+1)} \quad (0 < p < 1).$

879. Prove that $\Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin \pi a} \quad (0 < a < 1).$

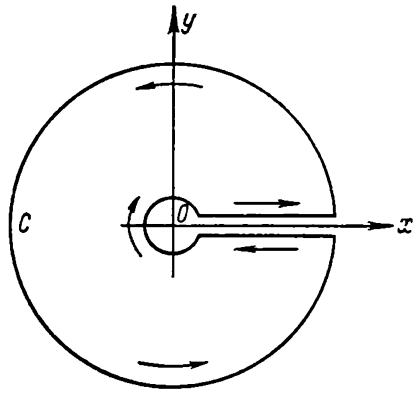

FIG. 18

HINT. Use the following known relation between the Beta and Gamma functions:

$$\Gamma(a)\Gamma(b) = \Gamma(a+b)B(a, b)$$

and in the integral defining the Beta function

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx,$$

change the variable, putting $x = y/(1+y)$.

REMARK. The relation proved in the problem only for real numbers a contained in the interval (0,1), holds for all complex numbers. For $z = -n$ where n is a natural number both parts of the equality become infinite. See, for example, [1, Chapter VII, § 4] or [3, Chapter VII, §1].

In problems **880-882** evaluate the given integrals.

880. $\displaystyle\int_0^\infty \frac{x^p\,dx}{1+x^2} \quad (-1 < p < 1).$

881. $\displaystyle\int_0^\infty \frac{x^p\,dx}{(1+x^2)^2} \quad (-1 < p < 3).$

882. $\displaystyle\int_0^\infty \frac{x^{-p}\,dx}{x^2+2x\cos\lambda+1} \quad (-1 < p < 1,\ -\pi < \lambda < \pi).$

883. On the positive real axis let the rational function $f(z)$ have only poles of the first order and let

$$\lim_{z\to 0}[z^p f(z)] = \lim_{z\to\infty}[z^p f(z)] = 0\,.$$

Prove that

$$\int_0^\infty x^{p-1}f(x)\,dx = \frac{\pi}{\sin p\pi}\sum_{k=1}^{n}\operatorname{res}\left[(-z)^{p-1}f(z)\right]_{z=\alpha_k} - \pi\cot p\pi\sum_{k=1}^{m}\beta_k^{p-1}\operatorname{res}\left[f(z)\right]_{z=\beta_k},$$

where the β_k are poles of the function $f(z)$ situated on the positive real axis, the α_k are other poles of $f(z)$ and the integral is taken in the sense of its principal value. (p is an integer.)

HINT. See problem **877**.

In problems **884**, **885** evaluate the principal values of the integrals.

884. $\int_0^\infty x^{p-1}\,dx/(1-x) \quad (0<p<1)$.

885. $\int_{-\infty}^{\infty} e^{px}\,dx/(1-e^x) \quad (0<p<1)$.

In problems **886**–**890** evaluate the given integrals.

886. $\int_0^1 \frac{x^{1-p}(1-x)^p}{(1+x)^3}\,dx \quad (-1<p<2)$.

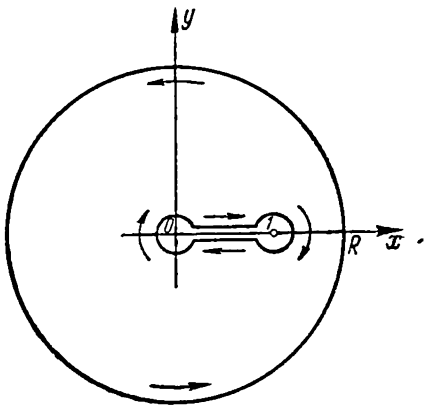

FIG. 19

HINT. Consider $\int_C \frac{z^{1-p}(1-z)^p}{(1+z)^3}\,dz$, where C is the contour indicated in Fig. 19 bounding a doubly connected domain, and pass to the limit as $R \to \infty$.

887. $\int_0^1 \frac{x^{1-p}(1-x)^p}{1+x^2}\,dx \quad (-1 < p < 2)$.

HINT. Prove that $\lim_{R\to\infty} \int_{C_R} \frac{z^{1-p}(1-z)^p}{1+z^2}\,dz = 2\pi i e^{-p\pi i}$, where C is the circle $|z| = R$ traversed in the positive direction.

888. $\int_0^1 \frac{x^{1-p}(1-x)^p}{(1+x)^2}\,dx \quad (-1 < p < 2)$.

889. $\int_{-1}^1 \frac{(1+x)^{1-p}(1-x)^p}{1+x^2}\,dx \quad (-1 < p < 2)$.

890. $\int_0^1 \frac{dx}{(x+1)\sqrt[3]{(x^2(1-x))}}$.

891. Evaluate the integral $\int_{-1}^1 \frac{dx}{(x-a)\sqrt{(1-x^2)}}$, where $\sqrt{(1-x^2)} > 0$ when $-1 < x < 1$, a is a complex number and $a \neq \pm 1$. In particular, find the values of the integral for $a = \pm e^{i\alpha}$ $(0 < \alpha < \pi)$, $a = iy$, $a < -1$ and $-1 < a < 1$ (the principal value).

892. Evaluate the integral $\int_0^1 \frac{x^{p-1}(1-x)^{-p}}{b-x}\,dx$, where $0 < p < 1$, b is a complex number and $b \neq 0$, $b \neq 1$.

In problems **893–895** find the given integrals.

893. $I_n = \int_0^1 \frac{x^{2n}dx}{(1+x^2)\sqrt{(1-x^2)}} \quad (n = 0, 1, 2, \ldots)$.

HINT. Put $x = 1/t$.

894. $I_n = \int_0^1 \frac{x^{2n}}{\sqrt[3]{(x(1-x^2))}} \quad (n = 0, 1, 2, \ldots)$.

895. $I_n = \int_0^1 \frac{dx}{\sqrt[n]{(1-x^n)}} \quad (n = 2, 3, \ldots)$.

HINT. Consider the integral $\int_C \frac{dz}{\sqrt[n]{(1+z^n)}}$ where C is the contour consisting of segments along the radius vectors of the points $1, \omega, \omega^2, \ldots, \omega^{n-1}$, where $\omega = e^{\frac{2\pi i}{n}}$, and along the circle $|z| = R$.

In problems **896–901** evaluate the integrals ($a \neq 0$ is a real number).

896. $\int_0^\infty \frac{\log x\,dx}{x^2+a^2}$.

HINT. Make use of the integral $\int_C \frac{\log z\,dz}{z^2+a^2}$, where the contour C is given in Fig. 20.

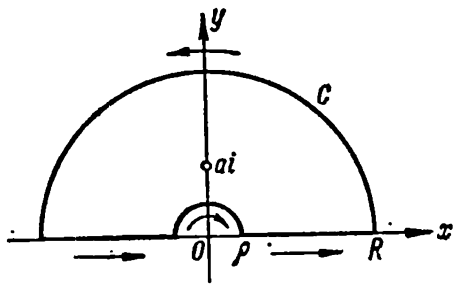

FIG. 20

897. $\int_0^\infty \frac{\log^2 x\,dx}{x^2+a^2}$.

898. $\int_0^\infty \frac{\log x\,dx}{(x^2+a^2)^2\sqrt{x}}$.

899. $\int_0^\infty \frac{\log x\,dx}{(x+1)^2\sqrt{x}}$.

900. $\int_0^1 \log\left(x+\frac{1}{x}\right)\frac{dx}{1+x^2}$.

901. $\int_1^\infty \log\left(\frac{e^x+1}{e^x-1}\right)\frac{dx}{x}$.

902. Let $f(z)$ be a rational function which has no poles on the positive part of the real axis or at the point $z = 0$, and is such that

$f(z) = O\left(\frac{1}{z}\right)$ as $z \to \infty$. Prove that

$$\int_0^\infty \frac{f(x)\,dx}{\log^2 x^2+\pi^2} = \sum_{k=1}^{n} \operatorname{res}\left[\frac{f(z)}{\operatorname{Log} z-\pi i}\right]_{z=a_k}$$

where $a_1 = -1$, and $a_2, a_3, \ldots, a_n$ are poles of the function $f(z)$ different from -1, and $\operatorname{Log} z = \log|z| + i \arg z$, $0 \leqslant \arg z < 2\pi$.

HINT. Consider the integral $\frac{1}{2\pi i}\int_C \frac{f(z)}{\operatorname{Log} z-\pi i}\,dz$ where the contour C is given in Fig. 21.

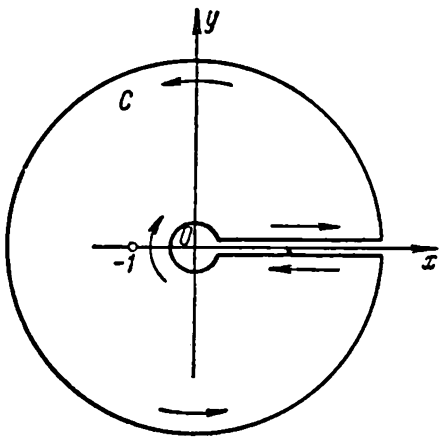

FIG. 21

In problems **903–905** evaluate the integrals assuming that $a > 0$ and n is a natural number.

903. (1) $\int_0^\infty \frac{dx}{(x+a)(\log^2 x+\pi^2)}$; (2) $\int_0^\infty \frac{dx}{(x^2+a^2)(\log^2 x+\pi^2)}$.

904. $\int_0^\infty \frac{dx}{(x^2+a^2)[\log^2 x+(2n+1)^2\pi^2]}$.

HINT. Make use of the integral

$$\int_C \frac{1}{z^2+a^2}\left[\frac{1}{\operatorname{Log} z-(2n+1)\pi i} + \frac{1}{\operatorname{Log} z-(2n-1)\pi i} + \ldots + \frac{1}{\operatorname{Log} z+(2n-1)\pi i}\right] dz$$

where the contour C is given in Fig. 21, and the branch of Log z is chosen as in problem **902**.

905. $\displaystyle\int_0^\infty \frac{dz}{(x^2+a^2)([\log x]^2+4n^2x^2)}$.

HINT. Make use of the integral

$$\int_C \frac{1}{z^2+a^2}\left[\frac{1}{\operatorname{Log} z-2n\pi i}+\frac{1}{\operatorname{Log} z-(2n-2)\pi i}+\ldots+\frac{1}{\operatorname{Log} z+(2n-2)\pi i}\right]dz,$$

where the contour C is given in Fig. 22, and the branch of Log z is chosen just as in problem **902**.

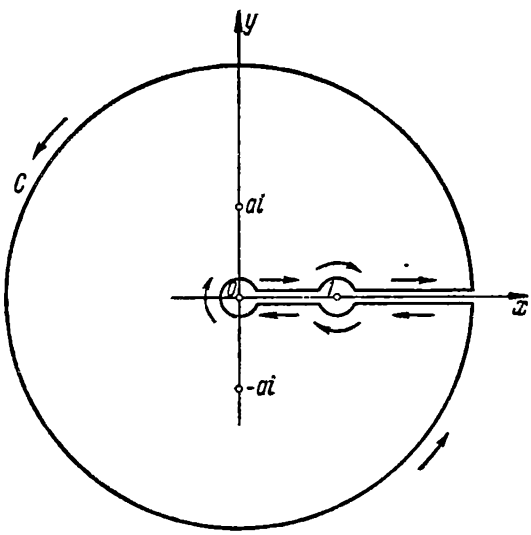

FIG. 22

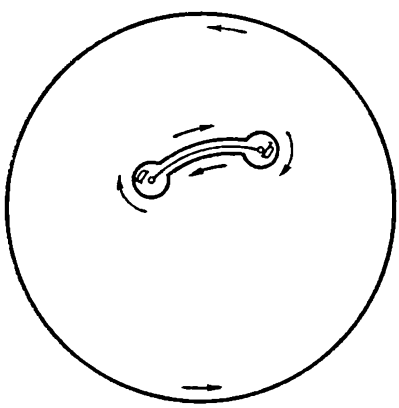

FIG. 23

906. Let $f(z)$ be a rational function which has no poles on the open contour C, the first point of whieh is a and last b. Prove that

$$\int_C f(z)\,dz = \sum \operatorname{res}\left[f(z)\operatorname{Log}\frac{z-b}{z-a}\right]+\operatorname{res}\left[f(z)\operatorname{Log}\frac{z-b}{z-a}\right]_{z=\infty}$$

where the summation extends over all the poles of the function $f(z)$

different from ∞ (the choice of the branch of the logarithm single-valued outside C is arbitrary).

HINT. Consider $\int_\Gamma f(z) \operatorname{Log} \frac{z-b}{z-a}\,dz$, where the contour Γ, bounding the doubly connected domain is given in Fig. 23.

In problems **907–910** find the given integrals, for real values of a.

907. $\int_0^\infty \frac{\sin ax\,dx}{\sinh x}$.

HINT. Make use of the integral $\int_C \frac{e^{aiz}dz}{\sinh z}$ where the contour C is given in Fig. 24.

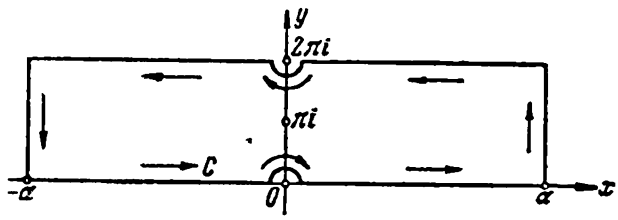

FIG. 24

908. $\int_0^\infty \frac{x\cos ax\,dx}{\sinh x}$.

909. $\int_0^\infty \frac{\cosh ax}{\cosh \pi x}dx \quad (-\pi < a < \pi)$.

HINT. Make use of the integral $\int_C \frac{e^{az}dz}{\cosh \pi z}$, where C is the perimeter of the rectangle $-\alpha \leqslant \operatorname{Re} z \leqslant \alpha$, $0 \leqslant \operatorname{Im} z \leqslant 1$.

910. $\int_0^\pi \frac{x\sin x\,dx}{1+a^2-2a\cos x} \quad (a > 0)$.

HINT. Make use of the integral $\int_C \frac{z\,dz}{a-e^{-iz}}$, where C is the perimeter of the rectangle $-\pi \leqslant \operatorname{Re} z \leqslant \pi$, $0 \leqslant \operatorname{Im} z \leqslant h$, and pass to the limit as $h \to \infty$.

From here to the end of this section it is assumed that $t > 0$, C_1 is the straight line $\operatorname{Re} z = \alpha > 0$, going from below upwards, α being chosen so that all the singularities of the integrand are situated on the left of C_1†.

911. Prove that if $f(z) \to 0$ as $\operatorname{Im} z \to \pm\infty$, $\alpha_1 \leqslant \operatorname{Re} z \leqslant \alpha_2$ and the function $f(z)$ is analytic in the strip $\alpha_1 \leqslant \operatorname{Re} z \leqslant \alpha_2$, then the integral $\int_C f(z)\mathrm{d}z$, where C is the straight line $\operatorname{Re} z = \alpha$, does not depend on the choice of α, if $\alpha_1 \leqslant \alpha \leqslant \alpha_2$.

912. Let the analytic function $\phi(z)$ have not more than a finite number of singularities $a_1, a_2, \ldots, a_n$ on the left of C_1 and let $\phi(z) \to 0$ as $z \to \infty$ and $\operatorname{Re} z \leqslant \alpha$.

Prove that

$$\frac{1}{2\pi i}\int_{C_1} \mathrm{e}^{zt}\phi(z)\,\mathrm{d}z = \sum_{k=1}^{n} \operatorname{res}\left[\mathrm{e}^{zt}\phi(z)\right]_{z=a_k}.$$

In problems **913–919** find the integrals (n is a natural number).

913. (1) $\dfrac{1}{2\pi i}\displaystyle\int_{C_1} \frac{\mathrm{e}^{zt}\mathrm{d}z}{z^{n+1}}$; (2) $\dfrac{1}{2\pi i}\displaystyle\int_{C_1} \frac{t^z\,\mathrm{d}z}{z^{n+1}}$ $(t^z = \mathrm{e}^{z\log t})$.

914. $\dfrac{1}{2\pi i}\displaystyle\int_{C_1} \frac{\mathrm{e}^{zt}\mathrm{d}z}{(z-a)^{n+1}}$.

915. $\dfrac{1}{2\pi i}\displaystyle\int_{C_1} \frac{\mathrm{e}^{zt}\mathrm{d}z}{z^2+1}$.

916. $\dfrac{1}{2\pi i}\displaystyle\int_{C_1} \frac{z\mathrm{e}^{zt}\mathrm{d}z}{z^2+1}$.

917. $\dfrac{1}{2\pi i}\displaystyle\int_{C_1} \frac{\mathrm{e}^{zt}}{z^2(z^2+1)}\mathrm{d}z$.

918. $\dfrac{1}{2\pi i}\displaystyle\int_{C_1} \frac{\mathrm{e}^{zt}\mathrm{d}z}{(z-a)(z-b)(z-c)}$.

919. $\dfrac{1}{2\pi i}\displaystyle\int_{C_1} \frac{t^z\,\mathrm{d}z}{z(z+1)\ldots(z+n)}$.

---

920. Using the identity $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin \pi z}$ (see the remark to

† The problems of this section are mainly connected with the inversion formula of the Laplace integral which plays an important part in the operational calculus. See, for example, [3, Chapter VI] or V. A. Ditkin and P. I. Kuznetsov, *Textbook on the Operational Calculus* (*Spravochnik po operatsionnomu ischisleniyu*), Gostekhizdat, 1951, (in this book there are also given extensive tables of formulae on the operational calculus).

problem **879**), prove that if $\operatorname{Re} \nu < 0$

$$\frac{1}{2\pi i} \int_{\gamma} \frac{e^z dz}{z^{\nu+1}} = \frac{1}{\Gamma(\nu+1)},$$

where the contour γ is given in Fig. 25.

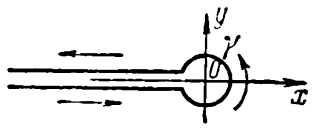

FIG. 25

REMARK. As the integral $\frac{1}{2\pi i} \int_{\gamma} \frac{e^z dz}{z^{\nu+1}}$ converges for $\operatorname{Re} \nu \geqslant 0$ also, it continues the function $\frac{1}{\Gamma(\nu+1)}$ analytically to the whole plane.

921. Prove that for $\operatorname{Re} \nu > -1$

$$\frac{1}{2\pi i} \int_{C_1} \frac{e^{zt} dz}{z^{\nu+1}} = \frac{t^\nu}{\Gamma(\nu+1)}.$$

In problems **922-933** find the given integrals.

922. (1) $\frac{1}{2\pi i} \int_{C_1} \frac{e^{zt} dz}{\sqrt{(1+z)}}$; (2) $\frac{1}{2\pi i} \int_{C_1} \frac{e^{zt} dz}{\sqrt{(z+i)}}$.

923. $\frac{1}{2\pi i} \int_{C_1} \frac{e^{zt} dz}{z\sqrt{(1+z)}}$.

924. $\frac{1}{2\pi i} \int_{C_1} \frac{e^{zt} dz}{(z+1)\sqrt{(z+2)}}$.

925. $\frac{1}{2\pi i} \int_{C_1} \frac{e^{zt}(1+e^{-\pi z})}{z^2+1} dz$.

926. $\frac{1}{2\pi i} \int_{C_1} \frac{e^{zt}(1-e^{-az})^2 dz}{z}$ $(a > 0)$.

927. $\frac{1}{2\pi i} \int_{C_1} \frac{e^{zt} dz}{z(1-e^{-az})}$ $(a > 0)$.

HINT. Make use of the expansion $\frac{1}{1-e^{-az}} = 1 + e^{-az} + e^{-2az} + \ldots$

928. $\frac{1}{2\pi i} \int_{C_1} \frac{e^{tz-x\sqrt{z}}}{z} dz$ $(x > 0)$.

HINT. Replace C_1 by the contour shown in Fig. 26.

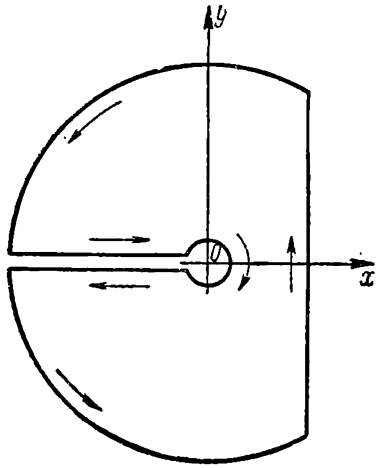

FIG. 26

929. $\dfrac{1}{2\pi i}\displaystyle\int_{C_1}\frac{e^{zt}\sinh r\sqrt{z}}{rz\sinh a\sqrt{z}}dz$ $(a>r>0)$.

930. $\dfrac{1}{2\pi i}\displaystyle\int_{C_1}\frac{\log(1+z)e^{zt}}{z}dz$.

931. $\dfrac{1}{2\pi i}\displaystyle\int_{C_1}e^{zt}\log\left(\frac{z+1}{z-1}\right)dz$.

932. $\displaystyle\int_0^\infty dt\int_{C_1}\frac{e^{z-\frac{at}{z}}}{z^2}dz$ $(a>0)$.

HINT. Change the order of integration.

933. $\dfrac{1}{2\pi i}\displaystyle\int_{C_1}\frac{e^{bz}}{z}dz\int_0^\infty e^{-az\cosh x}dx$ ($a>0$, b a real number).

HINT. Make use of the fact that $\displaystyle\int_{C_1}\frac{e^{-uz}}{z}dz=0$ for $u>0$.

934. From the series expansion of the Bessel function

$$J_\nu(z)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+\nu+1)}\left(\frac{z}{2}\right)^{\nu+2k},$$

deduce the following integral representations (γ is the contour given in problem **920**):

(1) $\dfrac{1}{2\pi i}\displaystyle\int_\gamma\frac{e^{\zeta-\frac{z^2}{4\zeta}}}{\zeta^{\nu+1}}d\zeta=\left(\frac{2}{z}\right)^\nu J_\nu(z)$;

(2) $\dfrac{1}{2\pi i}\displaystyle\int_{C_1}\frac{e^{\zeta-\frac{z^2}{4\zeta}}}{\zeta^{\nu+1}}d\zeta=\left(\frac{2}{z}\right)^{\nu}J_{\nu}(z)\ \ (\mathrm{Re}\,\nu>-1).$

HINT. Expand the function $e^{-\frac{z^2}{4\zeta}}$ into a power series and use the solutions of problems **920** and **921**.

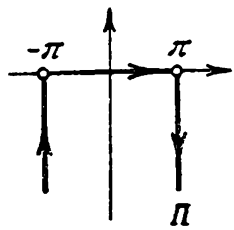

FIG. 27

935. Prove that if $\mathrm{Re}\,z>0$

$$J_{\nu}(z)=\frac{1}{2\pi}\int_{\Pi}e^{iz\sin\zeta-i\nu\zeta}d\zeta,$$

where Π is the contour given in Fig. 27, and hence obtain that for every integer n

$$J_n(z)=\frac{1}{\pi}\int_0^{\pi}\cos(z\sin\zeta-n\zeta)d\zeta.$$

In problems **936–938** find the integrals which involve Bessel functions.

936. $\displaystyle\int_0^{\infty}e^{-zt}J_n(t)\,dt$ ($\mathrm{Re}\,z>0$, n is an integer).

HINT. Use the integral representations of the preceding problem and change the order of integration.

937. (1) $\displaystyle\int_0^{\infty}J_0(at)\cos bt\,dt$; (2) $\displaystyle\int_0^{\infty}J_0(at)\sin bt\,dt$

(a and b are real numbers).

938. $\displaystyle\int_0^{\infty}\cos bx\,\frac{\sin t\sqrt{(x^2-a^2)}}{\sqrt{(x^2-a^2)}}dx$ $(t>|b|)$.

HINT. Make use of the fact that

$$\frac{\sin ut}{u} = \sqrt{\left(\frac{\pi t}{2u}\right)} J_{\frac{1}{2}}(ut) = \frac{\sqrt{(\pi t)}}{2} \frac{1}{2\pi i} \int_{C_1} \frac{e^{z - \frac{u^2 t^2}{4z}}}{z^{3/2}} dz$$

(see problem **934**), and change the order of integration.

The asymptotic behaviour of integrals†

939. Let the analytic function $\phi(z)$ have on the left of C_1 only a finite number of singularities, all of them being poles, and let $\phi(z) \to 0$ as $z \to \infty$ and $\operatorname{Re} z \leqslant \alpha$. Let us put

$$f(t) = \frac{1}{2\pi i} \int_{C_1} e^{zt} \phi(z) dz$$

Find $\lim_{t \to \infty} f(t)$. Consider various cases of the distribution of the poles with respect to the imaginary axis.

HINT. Use Jordan's lemma (see problem **411**).

940. Let the analytic function $\phi(z)$ have on the left of C_1 a finite number of singularities, and let $\phi(z) \to 0$ as $z \to \infty$ and $\operatorname{Re} z \leqslant \alpha$. Prove that the asymptotic equality

$$\frac{1}{2\pi i} \int_{C_1} e^{zt} \phi(z) dz \sim \sum \operatorname{res} [e^{zt} \phi(z)],$$

holds for large values of t, the summation extending over all the singularities of $\phi(z)$ with negative real part.

REMARK. The functions $f(t)$ and $F(t)$ are asymptotically equal as $t \to \infty$ ($f(t) \sim F(t)$), if $\lim_{t \to \infty} f(t)/F(t) = 1$.

941. Investigate the asymptotic behaviour as $t \to \infty$ of the function

$$f(t) = \frac{1}{2\pi i} \int_{C_1} \frac{e^{zt} dz}{z^2 (z+a)^3} \qquad (\operatorname{Re} a > 0).$$

† On the subject of this group of problems and also on the question of the applieation of asymptotic estimates and other methods for obtaining them see, for example, [3, Chapter V, § 3]; B. A. FUCHS and V. I. LEVIN, *Functions of a complex variable and some of their applications* (*Funktsii kompleksnogo peremennogo i nekotoryye ikh prilozheniya*), Chap. IV, Gostekhizdat, 1951 English translation published by Pergamon Press (1961); M. A. EVGRAFOV *Asymptotic estimates and integral functions* (*Asimptoticheskiye otsenki i tselyye funktsii*), Gostekhizdat, (1957).

942. Find an asymptotic expression as $t \to \infty$ for the function

$$f(t) = \frac{1}{2\pi i} \int_{C_1} \frac{ze^{zt-\sqrt{(z^2+2az)}}}{(z-\omega i)\sqrt{(z^2+2az)}}\,dz \qquad (\omega > 0,\ a > 0).$$

where $\sqrt{(z^2+2az)} > 0$ for $z > 0$.

HINT. Replace the contour C_1 by the contour represented in Fig. 26, and prove that the integrals along the arc of the circle and along the negative part of the real axis tend to zero as $t \to \infty$.

The series $\sum_{n=0}^{\infty} \frac{c_n}{z^n}$ is said to be the asymptotic expansion of the function $f(z)$ as $z \to \infty$, if

$$\lim_{z\to\infty} z^k \left[f(z) - \sum_{n=0}^{k} \frac{c_n}{z^n} \right] = 0 \qquad (k = 0, 1, 2, \ldots).$$

(From this it does not even follow that the series converges!)

Asymptotic expansions of a more general form are also frequently considered. Let $\{q_n(z)\}$ be an arbitrary sequence of functions such that $\lim_{z\to\infty} \frac{q_{n+1}(z)}{q_n(z)} = 0$, and $\{\mu_n(z)\}$ a sequence satisfying the conditions:

$$\lim_{z\to\infty} \frac{\mu_{n+1}(z)}{q_n(z)} = 0, \qquad \overline{\lim_{z\to\infty}} \left| \frac{\mu_n(z)}{q_n(z)} \right| > 0.$$

The series $\sum_{n=0}^{\infty} c_n\mu_n(z)$ is said to be an asymptotic expansion of the function

$$f(z) \sim \sum_{n=0}^{\infty} c_n\mu_n(z),$$

if

$$\lim_{z\to\infty} \frac{1}{q_k(z)} \left[f(z) - \sum_{n=0}^{k} c_n\mu_n(z) \right] = 0 \qquad (n = 0, 1, 2, \ldots).$$

Frequently the sequence $\left\{\frac{1}{z^{\alpha_n}}\right\}$ is chosen, as the sequence $\{\mu_n(z)\}$ where the α_n are positive real numbers monotonically tending to ∞.

943. Prove that for $x > 0$

$$\int_0^{\infty} \frac{e^{-xt}}{1+t^2}\,dt \sim \frac{1}{x} - \frac{2!}{x^3} + \frac{4!}{x^5} - \ldots + (-1)^n \frac{(2n)!}{x^{2n+1}} + \ldots$$

HINT. Make use of the expansion

$$\frac{1}{1+t^2} = 1-t^2+t^4- \ldots +(-1)^n t^{2n}+\frac{(-1)^{n+1}t^{2n+2}}{1+t^2}$$

and estimate the remainder term.

944. Prove that for $x > 0$

$$\int_h^\infty \frac{e^{x-t}}{t}\,dt \sim \frac{1}{x}-\frac{1}{x^2}+\frac{2}{x^3}- \ldots +(-1)^n\frac{n!}{x^{n+1}}+ \ldots$$

HINT. Integrate by parts and estimate the remainder.

945*. Prove that for $x > 0$

$$\int_{-x}^\infty \frac{e^{-x-t}}{t}\,dt \sim -\left(\frac{1}{x}+\frac{1}{x^2}+\frac{2}{x^3}+ \ldots +\frac{(n-1)!}{x^n}+ \ldots\right),$$

where the integral is understood as its principal value.

946. Prove that for real values of x

$$\int_{-\infty}^\infty \frac{e^{-t^2}dt}{t-x} \sim -\sqrt{\pi}\sum_{n=0}^\infty \frac{(2n)!}{2^{2n}.n!}\frac{1}{x^{2n+1}},$$

where the integral is understood as its principal value.

947. Prove that for real values of x

$$\int_0^\infty \frac{t^\alpha e^{-t^\beta}}{t-x}\,dt \sim -\frac{1}{\beta}\sum_{n=1}^\infty \Gamma\left(\frac{\alpha+n}{\beta}\right)\frac{1}{x^n} \qquad (\alpha > -1,\ \beta > 0)$$

where for $x > 0$ the integral is understood as its principal value.

948*. Prove that

$$\int_0^z e^{z^2-t^2}dt \sim \pm\frac{\sqrt{\pi}}{2}e^{z^2}-\left(\frac{1}{2z}-\frac{1}{2^2z^3}+\frac{1.3}{2^3z^5}-\frac{1.3.5}{2^4z^7}+ \ldots\right),$$

the signs $+$ or $-$ being taken according as $\operatorname{Re} z > 0$ or $\operatorname{Re} z < 0$. If $\operatorname{Re} z = 0$ the term in front of the brackets must be omitted.

949. Find the asymptotic expansion of the function

$$f(t)=\frac{1}{2\pi i}\int_{C_1}\frac{e^{zt}\sqrt{(z)}\,dz}{z^2+\omega^2}\qquad(\omega>0).$$

Find also the expansion of $f(t)$ for small t.

HINT. Replace C_1 by the contour given in Fig. 28. In order to obtain the asymptotic expansion of the integral $\int_0^\infty \frac{e^{-xt}\sqrt{x}\,dx}{x^2+\omega^2}$ use the hint to problem **943.** For small t it is necessary to choose C_1 such that α is greater than ω, and expand $1/(z^2+\omega^2)$ in a series.

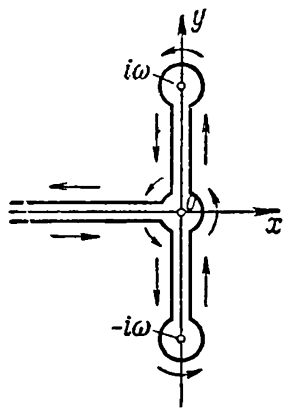

FIG. 28

950. Prove that

$$\frac{1}{2\pi i}\int_{C_1}\frac{e^{zt}}{\sqrt{(z)}(z^2+1)}dz-\sin\left(t-\frac{\pi}{4}\right)\sim\frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}(-1)^n\frac{(4n)!}{(2n)!}\left(\frac{1}{2\sqrt{t}}\right)^{4n+1}.$$

951. Find the asymptotic expansion of the function

$$f(t)=\frac{1}{2\pi i}\int_{C_1}\frac{e^{zt}dz}{z(1+z^{\frac{3}{2}})}\qquad(z^{\frac{3}{2}}>0 \text{ for } z>0).$$

Obtain also an approximate formula for $f(t)$ for small t.

HINT. In order to obtain the asymptotic expansion replace C_1 by the contour shown in Fig. 29, where $z_1=-\frac{1}{2}+\frac{i\sqrt{3}}{2}$, $z_2=-\frac{1}{2}-\frac{i\sqrt{3}}{2}$.

For small t the abscissa of the straight line C_1 must be taken greater than unity.

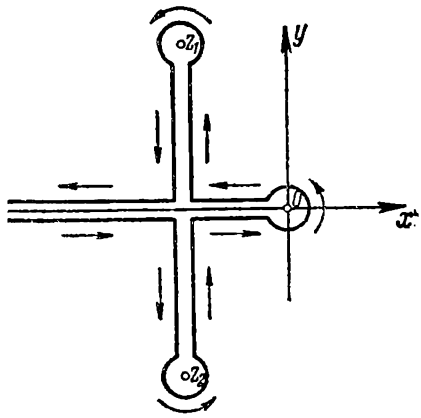

FIG. 29

§ 3. The distribution of zeros. The inversion of series

In problems **952–954** using Rouché's theorem find the number of roots of the given equations lying within the circle $|z| < 1$.

952. $z^9 - 2z^6 + z^2 - 8z - 2 = 0$.

953. $2z^5 - z^3 + 3z^2 - z + 8 = 0$.

954. $z^7 - 5z^4 + z^2 - 2 = 0$.

955. Prove that if the inequality

$$|a_k z^k| > |a_0 + a_1 z + \ldots + a_{k-1} z^{k-1} + a_{k+1} z^{k+1} + \ldots + a_n z^n|,$$

is satisfied at all the points of a contour C, then the polynomial

$$a_0 + a_1 z + \ldots + a_n z^n$$

has k zeros within the contour C, if the point $z = 0$ lies within this contour, and has no zeros if it lies outside the contour C.

956. How many roots of the equation

$$z^4 - 5z + 1 = 0$$

are situated inside the eircle $|z| < 1$? In the ring $1 < |z| < 2$?

957. How many roots of the equation

$$z^4 - 8z + 10 = 0$$

are situated inside the circle $|z| < 1$? In the ring $1 < |z| < 3$?

958. How many roots has the equation $z = \phi(z)$ in the circle $|z| < 1$, if for $|z| \leqslant 1$ the function $\phi(z)$ is analytic and satisfies the inequality $|\phi(z)| < 1$?

959. How many roots has the equation $e^z = az^n$ in the circle $|z| < R$ (n a natural number), if $|a| > e^R/R^n$?

960. Prove that in the right half-plane the equation

$$z = \lambda - e^{-z} \qquad (\lambda > 1)$$

has a unique (and hence real) root.

961*. Prove that no matter how small is $\varrho > 0$, for sufficiently great n all the zeros of the function

$$f_n(z) = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \ldots + \frac{1}{n!z^n}$$

are situated inside the circle $|z| < \varrho$.

962. Prove that if $\varrho < 1$ then the polynomial

$$P_n(z) = 1 + 2z + 3z^2 + \ldots + nz^{n-1}$$

for sufficiently great n has no zeros in the circle $|z| < \varrho$.

HINT. Use the method of solution of problem **961**.

963. The function $\phi(z)$ is meromorphic in the domain G and analytic on its boundary C. Prove the following assertions:

(1) If $|\phi(z)| < 1$ on C, then the number of roots of the equation $\phi(z) = 1$ situated inside the domain G is equal to the number of poles of the function $\phi(z)$ in the domain G.

(2) If $|\phi(z)| > 1$ on C, then the number of roots of the equation $\phi(z) = 1$ situated in the domain G is equal to the number of zeros of the function $\phi(z)$ in the domain G.

(3) The assertions (1) and (2) remain true if the equation $\phi(z) = 1$ is replaced by the equation $\phi(z) = \alpha$, where $|\alpha| \geqslant 1$ in case (1) and $|\alpha| \leqslant 1$ in case (2).

---

964. Let none of the zeros of the polynomial

$$P_n(z) = z^n + a_1 z^{n-1} + \ldots + a_n$$

lie on the imaginary axis.

Prove that when the point z traverses the imaginary axis from below upwards the increase of the argument of $P_n(z)$ equals $k\pi$, where k is an integer of the same parity as n, and $|k| \leqslant n$.

Prove that the polynomial $P_n(z)$ then has $(n+k)/2$ zeros in the right half-plane.

HINT. Represent $P_n(z)$ in the form

$$P_n(z) = z^n\left(1+\frac{a_1}{z}+\ \ldots\ +\frac{a_n}{z^n}\right)$$

and apply the principle of the argument to the semicircle $|z| < R$, $\operatorname{Re} z > 0$ for a sufficiently large R.

965. Find the number of zeros of the polynomial

$$z^6+z^5+6z^4+5z^3+8z^2+4z+1$$

in the right half-plane.

966. Find the number of roots of the equation

$$z^4+2z^3+3z^2+z+2 = 0$$

in the right half-plane and in the first quadrant.

967. How many roots in each quadrant has the equation

$$2z^4-3z^3+3z^2-z+1 = 0?$$

968. In which quadrants are the roots of the following equation to be found?

$$z^4+z^3+4z^2+2z+3 = 0.$$

969. Prove that the number of roots of the equation

$$z^{2n}+\alpha z^{2n-1}+\beta^2 = 0$$

(α and β are real numbers, $\alpha \neq 0$, $\beta \neq 0$; n is a natural number), which have positive real part is equal to n if n is even. However, if n is odd, the number of them is $n-1$ if $\alpha > 0$ and $n+1$ if $\alpha < 0$.

HINT. Consider the increase of $\arg(z^{2n}+\alpha z^{2n-1}+\beta^2)$ when the point z describes the boundary of a right semicircle of large radius.

If the coefficients of the polynomial

$$P_n(z) = z^n+a_1z^{n-1}+\ \ldots\ +a_{n-1}z+a_n$$

depend continuously on the real parameters α, β, then in order to find the relation between the number of zeros of $P_n(z)$ situated in the right half-plane and the parameters it is possible to proceed as follows (commencing with the fact that each zero depends continuously on the coefficients of the polynomial):

In the $\alpha\beta$-plane, construct the lines $P_n(i\tau) = 0$ (τ is a real parameter), that is, lines for the points of which among the roots of the polynomial there are purely imaginary (or zero) roots. These lines divide the $\alpha\beta$-plane into domains in each of which the number of zeros of $P_n(z)$ with positive real part is constant. This number can be found by taking an arbitrary point of the corresponding domain and applying to it, for example, the method of problem **964**.

In problems **970–972** determine the domains of the $\alpha\beta$-plane in which the number of zeros of the corresponding polynomial $P_n(z)$ with positive real part is constant; find this number m for each domain.

970. $P(z) = z^3+\alpha z^2+\alpha z+\beta$.

971. $P(z) = z^3+\alpha z^2+\beta z+1$.

972. $P(z) = z^3+(\alpha+\beta)z^2+(\alpha-\beta)z+\alpha$.

973. Let $f(z) = P_n(z)+Q_m(z)e^{-\tau z}$, where $\tau > 0$; $P_n(z)$ and $Q_m(z)$ are polynomials prime to one another, where $n > m$ and $f(z)$ has no zeros on the imaginary axis. N is the number of zeros of the polynomial $P_n(z)$ in the right half-plane. Prove that in order that the function $f(z)$ should have no zeros in the right half-plane it is necessary and sufficient that the point $w = -\dfrac{Q_m(z)}{P_n(z)}e^{-\tau z}$ should go round the point $w = 1$ in the positive direction N times while the point z traverses the whole imaginary axis from below upwards (if $P_n(z)$ has zeros on the imaginary axis it is necessary in the motion of the point z along this axis to avoid the zeros of $P_n(z)$ on the right by semicircles of sufficiently small radii).

In problems **974–976** it is necessary to find the domains in the space of the coefficients a, b (that is in the ab-plane) for which all the zeros of the corresponding function lie in the left half-plane, a and b are real numbers, and $\tau > 0$.

The use is recommended of the theorem of **973** and the method given on page 143.

974. $z+a+be^{-\tau z}$

975. $z^2+az+be^{-\tau z}$

976. $z^2+(az+b)e^{-\tau z}$.

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977. Prove by means of Rouché's theorem that if the function $w = f(z)$ has the expansion

$$f(z) = w_0+c_k(z-z_0)^k+\ldots \qquad (c_k \neq 0,\ k \geqslant 1)$$

in the neighbourhood of the point z_0 then, for sufficiently small $r > 0$, there exists a $\varrho > 0$ such that any value $w \neq w_0$ of the circle $|w-w_0| < \varrho$ is assumed precisely k times in the circle $|z-z_0| < r$ and moreover at distinct points.

978. Using the result of the preceding problem prove that an analytic function has the property of mapping domains onto domains.

979. Using the result of the preceding problem prove the maximum modulus principle for analytic functions. Prove that this principle is valid for arbitrary continuous mappings $w = f(z)$ which have the property described in **978**.

980. Prove that if in the conditions of problem **977** we have $k = 1$, that is, $f'(z_0) \neq 0$ then the function $f(z)$ establishes a one-one and conformal correspondence between some simply connected neighbourhood of the point z_0 and the small circle $|w-w_0| < \varrho$.

HINT. Consider the function $z = f^{-1}(w)$ in the small circle $|w-w_0| < \varrho$.

981. Prove that if $k > 1$ in the conditions of problem **977** then the function $w = f(z)$ maps one-one some simply connected neighbourhood of the point z_0 onto a k-sheeted circle with centre at the point w_0.

982. Extend the theorems proved in problems **980** and **981** to the cases when the point z_0 is a simple or multiple pole of the function $f(z)$.

983. Prove that if the expansion of the function $f(z)$ in the neighbourhood of infinity is of the form

$$f(z) = A_0 + \frac{A_1}{z} + \frac{A_2}{z^2} + \ldots + \frac{A_k}{z^k} + \ldots$$

then some neighbourhood of infinity can be mapped one-one and conformally onto a single-sheeted circle, if $A_1 \neq 0$, and onto a k-sheeted circle, if $A_1 = A_2 = \ldots = A_{k-1} = 0$ and $A_k \neq 0$.

REMARK. On the subject of this group of problems see, for example, [2, Chapter X].

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984. Let $F(z) = z-a-wf(z)$, the function $f(z)$ being analytic at the point $z = a$. Using Rouché's theorem prove that for sufficiently small $|w|$ there exists a circle K with centre at the point $z = a$, in which the function $F(z)$ has only one (simple) zero. Show also that if $f(a) \neq 0$, then for an appropriate choice of the value of w any point of some neighbourhood of the point $z = a$ can become a zero of the function $F(z)$.

985. Let $z = z(w)$ be a single-valued function, defined for sufficiently small $|w|$ by the equation $z-a-wf(z) = 0$, the function $f(z)$ analytic at the point $z = a$ and $f(a) \neq 0$. Prove that for every function $\Phi(z)$ analytic at the point $z = a$, for sufficiently small $|w|$ there holds the expansion

$$\frac{\Phi(z)}{1-wf'(z)} = \Phi(a) + \sum_{n=1}^{\infty} \frac{w^n}{n!} \frac{d^n}{da^n} \{\Phi(a)[f(a)]^n\}.$$

HINT. If we denote by C the circumference of the circle K, in which the equation $z-a-wf(z) = 0$ has only one root (see problem **984**), then

$$\frac{\Phi(z)}{1-wf'(z)} = \frac{1}{2\pi i} \int_C \frac{\Phi(\zeta)}{\zeta-a-wf(\zeta)} d\zeta.$$

Now expand the integrand into a series of powers of w and estimate the remainder term.

986. Using the notation of the preceding problem prove Lagrange's formula

$$\Phi(z) = \Phi(a) + \sum_{n=1}^{\infty} \frac{w^n}{n!} \frac{d^{n-1}}{da^{n-1}} \{\Phi'(a)[f(a)]^n\}.$$

Hence, in particular, obtain the expansion in a Taylor series of the function $z = z(w)$ itself.

HINT. Apply to the function $\phi(z)[1-wf'(z)]$ the solution of the preceding problem.

987. Expand in a series of powers of w each of the branches of the function $z(w)$, defined by the equation $w = 2z + z^2$ (for one branch $z(0) = 0$, for the other $z(0) = -2$).

988. Expand in a series of powers of w the branch of the function $z = z(w)$ defined by the equation $w = 2(z-a)/(z^2-1)$ for which $z(0) = a$.

989. Starting from the definition of the Legendre polynomials $P_n(z)$ in terms of the generating function $1/\sqrt{(1-2zt+t^2)}$ (see problem **522**), prove that $P_n(z) = \frac{1}{2^n n!} \frac{d^n}{dz^n} [(z^2-1)^n]$.

HINT. In the conditions of problem **988** apply Lagrange's formula to the function

$$\Phi(z) = \frac{1}{\sqrt{(1-2aw+w^2)}} = \frac{1-z^2}{z^2-2az+1}.$$

990. In the neighbourhood of the point $w = 0$ the function $z = z(w)$ is defined by the equation $w = ze^{-az}$. Expand in a series of powers of w:

(1) $z(w)$; (2) $e^{bz(w)}$.

991. Expand in powers of w the function $z = z(w)$, defined in the neighbourhood of the point $w = 0$ by Kepler's equation

$$z - a = w \sin z \qquad (a \neq 0, \pm\pi, \pm 2\pi, \ldots).$$

992*. Determine the radius of convergence obtained in the preceding problem for the expansion of $z(w)$ in the case when $a = \pi/2$.

993. Prove the following generalisation of Lagrange's theorem. Let $f(z)$ and $\phi(z)$ be functions analytic in the neighbourhood of the point a, C a circle with centre at the point a and radius r such that at all its points

$$|\alpha f(z) + \beta\phi(z)| < r.$$

If $\Phi(\zeta)$ is an analytic function of a single root of the equation

$$z - a - \alpha f(z) - \beta\phi(z) = 0$$

then

$$\Phi(\zeta) = \Phi(a) + \sum \frac{\alpha^m \beta^n}{m!\,n!} \frac{d^{m+n-1}}{da^{m+n-1}} \{\Phi'(a)[f(a)]^m [\phi(a)]^n\},$$

where the summation extends to all m and n except $m = n = 0$.

§ 4. Partial fraction and infinite product expansions. The summation of series

994. Let $f(z)$ be a meromorphic function with simple poles at the points $a_1, a_2, \ldots, a_n, \ldots$, where $0 < |a_1| \leqslant |a_2| \leqslant \ldots$ and $\lim_{n\to\infty} a_n = \infty$. Let us denote by A_n the residue of the function $f(z)$ at the pole a_n $(n = 1, 2, \ldots)$. Let us assume that there exists a sequence of closed contours C_m satisfying the following conditions:

(1) C_m does not pass through even one of the points a_n;

(2) Every contour C_m is contained within the contour C_{m+1};

(3) The minimum distance of the contour C_m from the origin of coordinates (let us denote it by R_m) increases without limit as $m \to \infty$;

(4) The ratio of the length L_m of the contour C_m to R_m remains bounded, that is, $L_m = O(R_m)$.

(5) $\max_{z \in C_m} |f(z)| = o(R_m)$ (condition 5 is obviously satisfied if the function $f(z)$ is bounded on all the contours C_m).

Prove that subject to these conditions the partial fraction expansion

$$f(z) = f(0) + \sum_{n=1}^{\infty} A_n \left(\frac{1}{z - a_n} + \frac{1}{a_n} \right),$$

of $f(z)$ is valid, the convergence of the series being uniform in every closed region, not containing points a_n, if terms referring to poles contained between C_m and C_{m+1} $(m = 1, 2, \ldots)$ are grouped under the summation sign.

HINT. Apply the residue theorem to the integral

$$\frac{1}{2\pi i} \int_{C_m} \frac{f(\zeta)\,d\zeta}{\zeta(\zeta - z)}$$

and pass to the limit as $m \to \infty$.

REMARK. For various generalisations of this theorem see, for example, [2, Chapter VIII, sec. 4].

In problems **995–1002** prove the validity of the expansions.

995. $\cot z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2\pi^2}$.

996. $\frac{1}{\sin z} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{(-1)^n 2z}{z^2 - n^2\pi^2}$.

997. $\tan z = \sum_{n=1}^{\infty} \frac{2z}{\left[\frac{(2n-1)\pi}{2}\right]^2 - z^2}$.

998. $\frac{1}{\cos z} = \pi \sum_{n=1}^{\infty} \frac{(-1)^n (2n-1)}{z^2 - \left[\frac{(2n-1)\pi}{2}\right]^2}$.

999. $\tanh z = \sum_{n=1}^{\infty} \frac{2z}{z^2 + \left[\frac{(2n-1)\pi}{2}\right]^2}$.

1000. $\frac{1}{\sinh z} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{(-1)^n 2z}{z^2 + n^2\pi^2}$.

1001. $$\frac{z}{e^z-1}=1-\frac{z}{2}+\sum_{n=1}^{\infty}\frac{2z^2}{z^2+4n^2\pi^2}.$$

1002. $$\frac{1}{\sin^2 z}=\sum_{n=-\infty}^{\infty}\frac{1}{(z-n\pi)^2}.$$

1003. Let $f(z)$ be an integral function with simple zeros at the points $a_1, a_2, \ldots, a_n, \ldots$, where

$$0<|a_1|\leqslant|a_2|\leqslant\ldots, \quad \lim_{n\to\infty} a_n=\infty.$$

Let us assume that there exists a system of contours $\{C_n\}$, satisfying conditions 1–4 of problem **994**, and on which

$$\max_{z\in C_m}\left|\frac{f'(z)}{f(z)}\right|=o(R_m).$$

Prove that the expansion

$$f(z)=f(0)e^{\frac{f'(0)}{f(0)}z}\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}}$$

is valid in the whole plane.

In problems **1004–1010** prove the validity of the expansions.

1004. $$\sin z=z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2\pi^2}\right).$$

1005. $$\cos z=\prod_{n=0}^{\infty}\left\{1-\left[\frac{2z}{(2n+1)\pi}\right]^2\right\}.$$

1006. $$\sinh z=z\prod_{n=1}^{\infty}\left(1+\frac{z^2}{n^2z^2}\right).$$

1007. $$\cosh z=\prod_{n=0}^{\infty}\left\{1+\left[\frac{2z}{(2n+1)\pi}\right]^2\right\}.$$

1008. $$e^z-1=ze^{\frac{z}{2}}\prod_{n=1}^{\infty}\left(1+\frac{z^2}{4n^2\pi^2}\right).$$

1009. $e^{az}-e^{bz} = (a-b)ze^{\frac{1}{2}(a+b)z}\prod_{n=1}^{\infty}\left[1+\frac{(a-b)^2z^2}{4n^2\pi^2}\right]$.

1010. $\cosh z-\cos z = z^2\prod_{n=1}^{\infty}\left(1+\frac{z^2}{4n^4\pi^4}\right)$.

1011. Let $f(z)$ be a meromorphic function with a finite number of poles: $a_1, a_2, \ldots, a_m$, not coincident with even one of the points $z = 0, \pm 1, \pm 2, \ldots$. Prove that if there exists a sequence of contours $\{C_n\}$, tending to the point at infinity, and

$$\lim_{n\to\infty}\int_{C_n} f(z)\cot \pi z\,dz = 0 \tag{1}$$

then

$$\sum_{n=-\infty}^{\infty} f(n) = -\pi\sum_{k=1}^{m}\operatorname{res}\left[f(z)\cot\pi z\right]_{z=a_k}.$$

1012. Prove that if in the conditions of the preceding problem the requirement (1) is replaced by the condition

$$\lim_{n\to\infty}\int_{C_n}\frac{f(z)\,dz}{\sin \pi z} = 0,$$

then

$$\sum_{n=-\infty}^{\infty}(-1)^n f(n) = -\pi\sum_{k=1}^{\infty}\operatorname{res}\left[\frac{f(z)}{\sin\pi z}\right]_{z=a_k}.$$

In problems **1013–1019** find the sum of the series, assuming the number a to be such that not one of the denominators becomes zero.

1013. $\sum_{n=-\infty}^{\infty}\frac{1}{(a+n)^2}$.

1014. $\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{(a+n)^2}$.

1015. $\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$.

1016. $\sum_{n=0}^{\infty}\frac{1}{n^2+a^2}$.

1017. $\sum_{n=0}^{\infty}\frac{(-1)^n}{n^2+a^2}$.

1018. $\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}$.

1019. $\sum_{n=1}^{\infty}(-1)^n \frac{n \sin bn}{a^2-n^2} \qquad (-\pi<b<\pi)$.

HINT. Mako use of the integral

$$\int_C \frac{ze^{ibz}dz}{(a^2-z^2)\sin \pi z}.$$

1020. Prove that

$$\sum_{n=1}^{\infty} \frac{n}{(n^2-3)\sqrt{(4n^2-3)}} = \int_0^{\sqrt{3}/2} \frac{x \cot \pi x}{(3-x^2)\sqrt{(3-4x^2)}}dx + \frac{1}{6}\cot[\pi(2-\sqrt{(3)}].$$

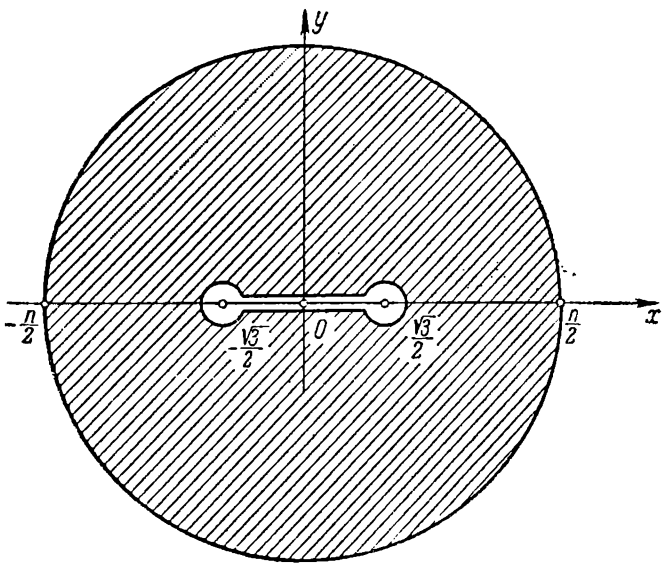

FIG. 30

HINT. Use the integral $\int_C \frac{z \cot \pi z \, dz}{(z^2-3)\sqrt{(4z^2-3)}}$, where C is the contour bounding the doubly connected domain shown in Fig. 30.

CHAPTER VIII

INTEGRALS OF CAUCHY TYPE. THE INTEGRAL FORMULAE OF POISSON AND SCHWARZ. SINGULAR INTEGRALS

§ 1. Integrals of Cauchy type

An integral of the form

$$\frac{1}{2\pi i}\int_C \frac{\phi(\zeta)\mathrm{d}\zeta}{\zeta - z},$$

where C is a smooth contour† (closed or open) and $\phi(\zeta)$ is a function, continuous on the contour C, with the exception perhaps of a finite number of points where it has an integrable discontinuity, is known as an *integral of Cauchy type*. The function $\phi(\zeta)$ is known as the *density* and $1/(\zeta - z)$ as the *kernel*. An integral of Cauchy type represents a function $F(z)$ which is analytic in every domain not containing points of the contour C. Hence

$$F^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{\phi(\zeta)\,\mathrm{d}\zeta}{(\zeta - z)^{n+1}}. \tag{1}$$

On the contour C let $\phi(\zeta)$ satisfy a Lipschitz condition of order α ($0 < \alpha \leqslant 1$ (for brevity, $\phi(\zeta) \in \mathrm{Lip}\,\alpha$), that is,

$$|\phi(\zeta_1) - \phi(\zeta_2)| < k|\zeta_1 - \zeta_2|^{\alpha},$$

where the points ζ_1 and ζ_2 belong to the contour C, and k is a constant. Then if a point of the contour ζ_0 is not an end point of it, there exists the *singular integral*

$$F(\zeta_0) = \frac{1}{2\pi i}\int_C \frac{\phi(\zeta)\,\mathrm{d}\zeta}{\zeta - \zeta_0}$$

defined as the principal value of the integral of Cauchy type.

† By a smooth contour we understand a simple (that is, without points of self-intersection) curve with continuously varying tangent and without cusps. However, the conditions imposed on the contour C can be significantly extended. See I. I. Privalov, (1950), *Boundary Properties of Analytic Functions* (*Granichnyye svoistva analiticheskikh funktsii*) 2nd ed., Chap. III. It is also possible to consider compound contours consisting of a finite number of contours of the given type.

This principal value can be expressed in terms of ordinary improper integrals by the formula

$$F(\zeta_0) = \frac{1}{2\pi i}\int_C \frac{\phi(\zeta)-\phi(\zeta_0)}{\zeta-\zeta_0}\,d\zeta + \frac{1}{2}\phi(\zeta_0) + \frac{\phi(\zeta_0)}{2\pi i}\operatorname{Log}\frac{b-\zeta_0}{a-\zeta_0}, \tag{2}$$

where the points a and b are the ends of the contour C, if it is open. The single-valued branch of Log is chosen so that for a closed contour ($a = b$) the term with the logarithm vanishes and the formula assumes the form

$$F(\zeta_0) = \frac{1}{2\pi i}\int_C \frac{\phi(\zeta)-\phi(\zeta_0)}{\zeta-\zeta_0}\,d\zeta + \frac{1}{2}\phi(\zeta_0). \tag{3}$$

If we denote by $F^+(\zeta_0)$ and $F^-(\zeta_0)$ the limiting values of the Cauchy type integral $F(z)$ as $z \to \zeta_0$ on the left of C and on the right of C respectively, then by Sokhotskii's formulae

$$\left.\begin{aligned} F^+(\zeta_0) &= F(\zeta_0) + \frac{1}{2}\phi(\zeta_0) \\ F^-(\zeta_0) &= F(\zeta_0) - \frac{1}{2}\phi(\zeta_0) \end{aligned}\right\} \tag{4}$$

or

$$F(\zeta_0) = \frac{1}{2}[F^+(\zeta_0)+F^-(\zeta_0)], \quad F^+(\zeta_0)-F^-(\zeta_0) = \phi(\zeta_0). \tag{5}$$

If the contour C is closed and is traversed in the usual direction, then $F^+(\zeta)$ is the limiting value of the function $F^+(z)$, determined within the contour (the domain D^+), and $F^-(\zeta)$ is the function determined outside the contour (the domain D^-)†. (See, for example, [1, Chapter III, § 3] or [3, Chapter III, § 3]).

1021. Prove the formula (1) for $n = 1$, find the limit of the difference

$$\frac{F(z+h)-F(z)}{h} - \frac{1}{2\pi i}\int_C \frac{\phi(\zeta)\,d\zeta}{(\zeta-z)^2}$$

as $h \to 0$.

1022. Prove formula (1) for any n by the method of mathematical induction.

1023. Prove that if C is a closed contour and the density of the Cauchy type integral

$$\Phi(z) = \frac{1}{2\pi i}\int_C \frac{\phi(\zeta)}{\zeta-z}\,d\zeta$$

can be represented in the form

$$\phi(\zeta) = \phi^+(\zeta)+\phi^-(\zeta)$$

† The case when the contour is an infinite curve dividing the plane into two domains is also referred to this one.

where $\phi^+(\zeta)$ and $\phi^-(\zeta)$ are the boundary values of functions analytic respectively inside and outside the contour C, then

$$\Phi^+(z) = \phi^+(z), \qquad \Phi^-(z) = -\phi^-(z) + \phi^-(\infty).$$

REMARK. If in the conditions of the problem one of the functions ϕ^- or ϕ^+ is identically equal to zero the Cauchy type integral becomes the Cauchy integral of the interior or exterior domain respectively.

1024. Let C be a closed contour. Find $F^+(z)$ and $F^-(z)$, if the density of the Cauchy type integral is the function (n is a natural number):

(1) $\phi(\zeta) = (\zeta - a)^n$;

(2) $\phi(\zeta) = \dfrac{1}{(\zeta - a)^n}$ (a is inside C);

(3) $\phi(\zeta) = \dfrac{1}{(\zeta - a)^n}$ (a is outside C).

1025. Find $F^+(z)$ and $F^-(z)$, if:

(1) The function $\phi(\zeta)$ is the boundary value of a function analytic in D^+, with the exception of a finite number of points a_k, where it has poles;

(2) The function $\phi(\zeta)$ is the boundary value of a function analytic in D^-, with the exception of a finite number of points a_k, where it has poles (among the points a_k there may also occur the point $z = \infty$).

1026. Find $F^+(z)$ and $F^-(z)$, if

$$\phi(\zeta) = \frac{\zeta^3 + i\zeta^2 - \zeta + 4i}{\zeta^4 - 3\zeta^2 - 4} + \frac{\log[(\zeta - 2)/(\zeta - 3)]}{\zeta^2 - 4}$$

and C is the circle $|\zeta| = 3/2$.

1027. Find $F^+(z)$ and $F^-(z)$, if $\phi(\zeta) = \cot \zeta$ and C is the circle $|\zeta| = 5$.

1028. Find $F^+(z)$ and $F^-(z)$, if $\phi(\zeta) = \zeta^2/(\zeta^2 + 1)$ and C is the real axis traversed from left to right.

REMARK. By an integral of Cauchy type taken along the real axis

$$F(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\phi(\tau)}{\tau - z} d\tau,$$

is to be understood its principal value if it diverges in the usual sense.

1029. Find $F^+(z)$ and $F^-(z)$, and also the limiting values of $F^{\pm}(\zeta)$ on the contour of integration C, if C is the circle $|\zeta| = R$, and

$$\phi(\zeta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)$$

is the uniformly convergent Fourier series of the real function $\psi(\theta) = \phi(Re^{i\theta})$.

1030. (1) Let C be the circle $|\zeta| = \pi/2$ and $f(\zeta)$ a function analytic in the circle $|\zeta| \leqslant \pi/2$. Find the functions defined by the integrals

$$I_1(z) = \frac{1}{2\pi i} \int_C f(\zeta) \cot(\zeta - z)\, d\zeta,$$

$$I_2(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)\, d\zeta}{\sin(\zeta - z)},$$

in domains, the points z of which possess the property that not one of the points $z + k\pi$ (k is an integer) lies on C.

(2) Solve the problems formulated in part (1) on the assumption that C is the circle $|\zeta| = \pi$.

1031. Let C be the segment $[-1, 1]$, traversed from left to right, and $\phi(\zeta) \equiv 1$. Find $F(z)$ outside C, the limiting values $F^{\pm}(\zeta)$ and the principal value of $F(\zeta)$ on C. Evaluate in particular $F(\pm i), F^{\pm}(0)$ and $F(0)$.

1032. Let C be the semicircle $|\zeta| = R$, $0 < \arg \zeta < \pi$ (commencing at the point R) and $\phi(\zeta) \equiv 1$. Find $F(z)$ outside C, the limiting values $F^{\pm}(\zeta)$ and the principal value of $F(\zeta)$ on C. Evaluate, in particular, $F(0)$, $F^{\pm}(iR)$ and $F(iR)$. Find also $F'(0)$.

1033. Let C be the semieircle $|\zeta| = R$, $-\pi < \arg \zeta < 0$ (commencing at the point R) and $\phi(\zeta) \equiv 1$. Find $F(z)$ outside C, the limiting values of $F^{\pm}(\zeta)$ on C, $F(0)$ and $F'(0)$.

1034. Let the density of a Cauchy type integral $\phi(\zeta) = 1/\zeta^n$. Find $F(z)$ outside C, if the eontour C is given by:

(1) The boundary of the ring $r < |z| < R$;

(2) The straight line $\operatorname{Im} \zeta = \pi$, traversed from left to right;

(3) The boundary of the strip $|\operatorname{Im} z| < \pi$;

(4) The semicircle $|\zeta| = R$, $0 < \arg \zeta < \pi$ (commencing at the point R);

(5) The semicircle $|\zeta| = R$, $-\pi < \arg \zeta < 0$ (commencing at the point R).

In parts (4) and (5) find the limiting values of $F^{\pm}(\zeta)$ on C and evaluate $F(0)$.

In problems **1035–1040** find $F(z)$ outside C, assuming that the contour C is an arc connecting the points a and b, and $\phi(\zeta)$ is the given function.

1035. $\phi(\zeta) \equiv 1$.

1036. $\phi(\zeta) = \zeta$.

1037. $\phi(\zeta) = \zeta^n$.

1038. $\phi(\zeta) = \sum_{n=0}^{\infty} c_n \zeta^n$, an integral function.

1039. $\phi(\zeta) = 1/(\zeta - z_0)$ $(z_0 \overline{\in} C)$.

1040. $\phi(\zeta) = 1/(\zeta - z_0)^n$ $(z_0 \overline{\in} C)$. Evaluate, in particular, $F(z_0)$.

HINT. Use the equality

$$\frac{\phi(\zeta)}{\zeta - z} = \frac{\phi(\zeta) - \phi(z)}{\zeta - z} + \frac{\phi(z)}{\zeta - z}.$$

1041. Find $F^+(z)$, $F^-(z)$ and the limiting values $F^{\pm}(\zeta)$ if C is the circle $|\zeta| = R$ and $\phi(\zeta)$ is the logarithmic function defined by the conditions:

(1) $\phi(\zeta) = \log \zeta = \log R + i\phi, \quad -\pi < \phi \leqslant \pi;$

(2) $\phi(\zeta) = \operatorname{Log} \zeta = \log R + i\phi, \quad 0 \leqslant \phi < 2\pi.$

HINT. Consider the contour consisting of the circle $|z| = R$ with a cut along the radius $[-R, 0]$ in the first case and $[0, R]$ in the second.

REMARK. If the branch of the logarithm is not fixed in advance and its continuous prolongation along the contour of integration is required, then the integral will depend on the choice of the initial point of integration. See also the examples given on page 65.

1042. Find $F^+(z)$, $F^-(z)$ and the limiting values $F^{\pm}(\zeta)$ on C, if $\phi(\zeta) = \log\left(\zeta/(\zeta-1)\right)$, and the contour C is:

(1) The circle $|\zeta| = R$ $(R > 1)$;

(2) The straight line $\operatorname{Im} \zeta = 1$, traversed from left to right.

1043. Find $F(z)$ and the limiting values $F^{\pm}(\zeta)$ on C, if $\phi(\zeta) = \log\left(\zeta/(\zeta - 1)\right)$, and the contour C is the semicircle $|\zeta| = R (R > 1)$, $0 \leqslant \arg \zeta \leqslant \pi$ (commencing at the point R).

1044. Find $F^+(z)$ and $F^-(z)$, if $\phi(\zeta) = \sqrt{\zeta}$ $(0 \leqslant \arg \sqrt{\zeta} < \pi)$ and C is the circle $|\zeta| = 1$.

In problems **1045–1047** find $F^{\pm}(z)$, if C is a closed contour, the points a and b lie within it and $\phi(\zeta)$ is a single-valued branch of a many-valued function defined outside the cut which connects the points a and b and lies within the contour C.

1045. $\phi(\zeta) = \operatorname{Log}\left((\zeta - a)/(\zeta - b)\right)$ $(\operatorname{Log} 1 = 0)$.

1046. $\phi(\zeta) = \sqrt{[(\zeta-a)/(\zeta-b)]}$ $(\phi(\infty)=+1)$.

HINT. In order to find the integral along the contour surrounding the cut, expand $\frac{\phi(\zeta)}{\zeta-z}$ in a series of powers of $\frac{1}{\zeta}$.

1047. $\phi(\zeta) = (\zeta-a)^{\lambda}(\zeta-b)^{1-\lambda}$ $\left(\left.\frac{\phi(\zeta)}{\zeta-b}\right|_{\zeta=\infty} = 1\right)$.

1048. Find $F^{\pm}(z)$ if the contour C is closed, the point a belongs to the domain D^+, the point b to the domain D^- and $\phi(\zeta) = \operatorname{Log}[(\zeta-a)/(\zeta-b)]$ $(\operatorname{Log} 1 = 0)$ is the single-valued branch defined outside the cut connecting the points a and b and intersecting the contour C at one point ζ_0.

HINT. Join to C the cut along the arc $\zeta_0 a$, if z is in the domain D^+, and that along $\zeta_0 b$ if z is in the domain D^-.

1049. Prove the validity of the equalities $(0 < \lambda < 1)$:

$$(1)\quad \int_0^1 \left(\frac{t}{1-t}\right)^{\lambda} \frac{dt}{t-z} = \frac{\pi}{\sin \lambda\pi}\left[1-\left(\frac{z}{z-1}\right)^{\lambda}\right] \qquad (z \overline{\in} [0, 1]);$$

$$(2)\quad \int_0^1 \left(\frac{t}{1-t}\right)^{\lambda} \frac{dt}{t-\tau} = \frac{\pi}{\sin \lambda\pi}\left[1-\left(\frac{\tau}{1-\tau}\right)^{\lambda}\cos \lambda\pi\right] \qquad (\tau \in (0, 1)).$$

HINT. In order to obtain the first formula consider the Cauchy integral

$$\frac{1}{2\pi i}\int_C \left(\frac{\zeta}{\zeta-1}\right)^{\lambda} \frac{d\zeta}{\zeta-z},$$

where $\left(\frac{\zeta}{\zeta-1}\right)^{\lambda}$ is a function single-valued in the plane with the straight line cut [0,1] and equal to 1 at ∞, and C is the boundary of the doubly connected domain: the disk $|\zeta| < R$ $(R > 1)$ with a cut along the segment [0,1]. The second formula is obtained from the first by means of Sokhotskii's formulae.

1050. Evaluate the singular integral

$$\frac{1}{\pi}\int_{-1}^{1} \sqrt{\left(\frac{1-t}{1+t}\right)}\frac{t^2+3}{t-x}\,dt \qquad (-1 < x < 1).$$

1051. Find the integrals:

$$(1)\quad \int_0^1 \log\frac{1-t}{t}\frac{dt}{t-z} \qquad (z \overline{\in} [0, 1]);$$

$$(2)\ \int_0^1 \log\frac{1-t}{t}\frac{dt}{t-\tau} \qquad (\tau\in(0,1))\,.$$

HINT. Use the method recommended in problem **1049** and the limit

$$\lim_{\lambda\to 0}\frac{\left(\frac{t}{1-t}\right)^{\lambda}-1}{\lambda}=\log\frac{t}{1-t}\,.$$

1052. Consider the singular integral

$$F(z,t_0)=\frac{1}{2\pi i}\int_C\frac{d\tau}{(\tau-t_0)^{\gamma}(\tau-z)} \qquad (\gamma=\alpha+i\beta,\ 0\leqslant\alpha<1)\,,$$

where C is an arc connecting the points a and b, t_0 a point of this arc and $(\tau-t_0)^{\gamma}$ is a single-valued branch in the plane with a cut joining the points t_0 and ∞. If the point t_0 coincides with one of the ends of the arc C, we shall consider that the cut extends along the whole of the contour C; if, however, the point t_0 is internal, the cut extends along the arc (t_0, b) of the contour C.
Prove the following assertions:

(1) In the neighbourhood of the point a

$$F(z,a)=\frac{e^{i\gamma\pi}}{2i\sin\gamma\pi}(z-a)^{-\gamma}+F_1(z) \qquad (z\overline{\in}C),$$

$$F(t,a)=\frac{\cot\gamma\pi}{2i}(t-a)^{-\gamma}+F_1(t) \qquad (t\in C),$$

where $F_1(z)$ is a function analytic in the neighbourhood of the point a.

HINT. Consider the difference

$$F_1(z)=\frac{1}{2\pi i}\int_C\frac{d\tau}{(\tau-a)^{\gamma}(\tau-z)}-\frac{e^{i\gamma\pi}}{2i\sin\gamma\pi}(z-a)^{-\gamma}$$

and using Sokhotskii's formulae prove that $F_1(z)$ is an analytic function in the neighbourhood of the point a.

(2) In the neighbourhood of the point b

$$F(z,b)=-\frac{e^{-i\gamma\pi}}{2i\sin\gamma\pi}(z-b)^{-\gamma}+F_2(z) \qquad (z\overline{\in}C),$$

$$F(t, b) = -\frac{\cot \gamma\pi}{2i}(t-b)^{-\gamma}+F_2(t) \qquad (t \in C),$$

where $F_2(z)$ is a function analytic in the neighbourhood of the point b.

(3) In the neighbourhood of an interior point t_0 of the contour C

$$F(z, t_0) = (z-t_0)^{-\gamma}+F_3(z) \quad \text{on the left of } C,$$

$$F(z, t_0) = F_4(z) \quad \text{on the right of } C,$$

$$F(t, t_0) = \frac{1}{2}(t-t_0)^{-\gamma}+F_5(z) \quad \text{if } t \in C,$$

where $F_3(z)$, $F_4(z)$ and $F_5(z)$ are analytic in the neighbourhood of the point t_0.

1053. Explain the behaviour of the Cauchy type integral

$$\frac{1}{2\pi i}\int_C \log\frac{\zeta}{\zeta-1}\frac{d\zeta}{\zeta-z}$$

close to the points $z = -R$ and $z = R$ if C is the semicircle $|\zeta| = R$ $(R > 1)$, in the upper half-plane (commencing at the point R).

HINT. See problem **1043.**

REMARK. For the behaviour of integrals of Cauchy type close to a singular line see N. I. MUSHKELISHVILI, (1951), *Singular Integral Equations* (*Singulyarnyye integral'nyye uravneniya*) Chap. I, Gostekhizdat, or F. D. GAKHOV, (1958), *Boundary Problems* (*Kraevyye zadachi*) Chap. I, Fizmatgiz.

§ 2. Some integral relations and double integrals

In the problems of this seetion the following notation is used: G is the domain bounded by the contour C,

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right),$$

$$d\bar{z} = dx - i dy = \overline{dz}, \qquad d\phi = \frac{\partial \phi}{\partial z}dz + \frac{\partial \phi}{\partial \bar{z}}d\bar{z},$$

(see page 55)

1054†. Using Green's formula prove the following relations:

† For problems **1054–1057** see the supplement by M. Schiffer to the book: R. COURANT, *Dirichlet's Principle, Conformal Mapping and Minimal Surfaces.* Interscience, New York, 1950.

$$(1)\ \iint_G \frac{\partial f}{\partial \bar{z}}\,dx\,dy = \frac{1}{2i}\int_C f\,d\zeta;$$

$$(2)\ \int_C f\,d\zeta + g\,d\bar{\zeta} = 2i\iint_G \left(\frac{\partial f}{\partial \bar{z}} - \frac{\partial g}{\partial z}\right)dx\,dy;$$

$$(3)\ f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)\,d\zeta}{\zeta - z} - \frac{1}{\pi}\iint_G \frac{\partial f(\zeta)}{\partial \bar{\zeta}}\,\frac{d\xi\,d\eta}{\zeta - z} \quad (\zeta = \xi + i\eta).$$

HINT. Make use of the fact that $\frac{\partial f(\zeta)}{\partial \bar{\zeta}}\,\frac{1}{\zeta - z} = \frac{\partial}{\partial \bar{\zeta}}\left(\frac{f(\zeta)}{(\zeta - z)}\right)$, and apply formula (1) to the domain G excluding the small circle $|\zeta - z| < \varrho$ $(\varrho \to 0)$.

1055. Assuming that the functions f and g are continuously differentiable in the closed region $\overline{G}$, prove that the expression $f\,dz + g\,d\bar{z}$ is the total differential of some function when and only when

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial g}{\partial z}.$$

1056. Prove that if f and g are functions analytic in the domain G and continuously differentiable on C, then

$$\iint_G f'\bar{g}\,dx\,dy = \frac{i}{2}\int_C f\bar{g}\,d\bar{\zeta},$$

in particular

$$\iint_G f'\bar{g}'\,dx\,dy = \frac{i}{2}\int_C f\,d\bar{g}.$$

1057. Prove that if the function $f(z)$ conformally maps the domain G onto a domain G', bounded by a contour $C' = f(C)$, then the integral

$$I = \frac{i}{2}\int_C f\,d\bar{f}$$

is equal to the area S of the domain G'; if, however, $f(z)$ conformally maps the domain G onto the exterior of the contour C', then $I = -S$.

1058†. Prove that if the function $\phi(\zeta)$ is defined and continuous in the bounded closed region $\bar{G}$, then the function

$$F(z) = \iint_G \frac{\phi(\zeta)\,d\xi\,d\eta}{\zeta - z} \qquad (\zeta = \xi + i\eta);$$

(1) Is analytic outside G and $F(\infty) = 0$.
(2) Is continuous in the whole plane and

$$|F(z_2) - F(z_1)| < M\,|z_2 - z_1|\,|\log \alpha\,|z_2 - z_1||\,,$$

where M and α are constants, and z_1 and z_2 are arbitrary points of the region $\bar{G}$.

HINT. In (2), to estimate the integral $\iint_G \frac{d\xi d\eta}{|\zeta - z_2|\,|\zeta - z_1|}$ make use of the fact that this integral is obviously less than

$$\iint_{\left|\zeta - \frac{z_1+z_2}{2}\right| < R} \frac{d\xi d\eta}{|\zeta - z_2|\,|\zeta - z_1|}$$

where R is the radius of the circle with centre at the point $(z_1+z_2)/2$, which contains within it the closed region $\bar{G}$. By a simple substitution we arrive at the last integral in the form

$$\iint_{|\zeta|<R} \frac{d\xi d\eta}{|\zeta - a|\,|\zeta + a|} = 2 \iint_{\substack{|\zeta|<R \\ \xi>0}} \frac{d\xi d\eta}{|\zeta - a|\,|\zeta + a|}, \qquad \text{where } a = \frac{|z_2 - z_1|}{2}.$$

From geometrical considerations it is clear that

$$\iint_{\substack{|\zeta|<R \\ \xi>0}} \frac{d\xi d\eta}{|\zeta - a|\,|\zeta + a|} < \iint_{|\zeta - a|<a} \frac{d\xi d\eta}{|\zeta - a|\,|\zeta + a|} + \iint_{\substack{a<|\zeta - a|<R/2 \\ \xi>0}} \frac{d\xi d\eta}{|\zeta - a|\,|\zeta + a|}.$$

Noting that in the first integral $|\zeta + a| > a$ and in the second $|\zeta + a| > |\zeta - a|$, and passing to polar coordinates $(\zeta - a = re^{i\phi})$, we obtain the required estimate.

1059. Let G be the circle $|z| < R < 1$ and

$$\phi(\zeta) = \frac{e^{2i\theta}}{\log \dfrac{1}{\varrho}} \qquad (\zeta = \varrho e^{i\theta},\ z = re^{i\phi}).$$

† For problems **1058–1062** see I. N. VEKUA, *Generalised Analytic Functions* (*Obobshchennyye analiticheskiye funktsii*), Fizmatgiz, 1959. English translation published by Pergamon Press (1962).

Find the function

$$F(z) = -\frac{1}{\pi}\iint_G \frac{\phi(\zeta)\,d\xi\,d\eta}{\zeta - z} \qquad (z \in G)$$

and its partial derivates $\partial F/\partial x$, $\partial F/\partial y$, $\partial F/\partial \bar{z}$, $\partial F/\partial z$ for $z \neq 0$. Show that of the given derivatives only $\partial F/\partial \bar{z}$ exists and is continuous at the coordinate origin.

1060. Prove that if the function $f(z)$ is continuously differentiable, then

$$\frac{\partial f}{\partial \bar{z}} = \lim \frac{\frac{1}{2i}\int_C f\,d\zeta}{\iint_G d\xi\,d\eta},$$

when the domain G shrinks to the point z.

HINT. Use the relation (1) of problem **1054.**

1061. Let the function $\phi(\zeta)$ be continuous in the closed region $\overline{G}$. Prove that the function

$$F(z) = -\frac{1}{\pi}\iint_G \frac{\phi(\zeta)\,d\xi\,d\eta}{\zeta - z}$$

satisfies the equation

$$\frac{\partial F}{\partial \bar{z}} = \phi(z)$$

in the domain G if the derivative $\partial F/\partial \bar{z}$ is defined by the formula of problem **1060.**

1062. Prove that with the conditions of problem **1061** the general solution of the equation

$$\frac{\partial f}{\partial \bar{z}} = \phi(z)$$

in the domain G can be represented in the form

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)\,d\zeta}{\zeta - z} - \frac{1}{\pi}\iint_G \frac{\phi(\zeta)\,d\xi\,d\eta}{\zeta - z} \quad \text{(Pompeius's formula).}$$

HINT. Use the relation (3) of problem **1054.**

§ 3. Dirichlet's integral, harmonic functions, the logarithmic potential and Green's function

The integral

$$D(u) = \iint\limits_G (u_x^2+u_y^2)\,dx\,dy$$

is known as *Dirichlet's integral*, and the integral

$$D(u, v) = \iint\limits_G (u_x v_x+u_y v_y)\,dx\,dy$$

is its corresponding *bilinear form*.

1063. Prove the following properties of Dirichlet's integral and its corresponding bilinear form:

(1) $D(u) = \iint\limits_G \left(u_r^2+\frac{1}{r}u_\phi^2\right)dx\,dy \qquad (z = x+iy = re^{i\phi})$;

(2) $D(u)$ and $D(u, v)$ are invariants in conformal mappings of the domain G;

(3) $D^2(u,v) \leqslant D(u)\,D(v)$ (the sign of equality holds only for the case $u/v =$ constant);

(4) If the function $f(z) = u+iv$ is analytic in the domain G, then

$$D(u) = D(v) = D(f) = \iint\limits_G |f'|^2\,dx\,dy.$$

In this case what is the geometrical meaning of $D(f)$?

1064. Using Green's formula

$$\iint\limits_G v\Delta u\,dx\,dy+D(u, v) = \int\limits_C v\frac{\partial u}{\partial n}\,ds$$

(n is the outward normal; Δ is Laplace's operator), prove the following properties of the harmonic functions u, u_1, u_2 (v is the function conjugate to u):

(1) $D(u) = \int\limits_C u\frac{\partial u}{\partial n}\,ds = \int\limits_C u\,dv$;

(2) $\int\limits_C \frac{\partial u}{\partial n}\,ds = \int\limits_C dv = 0$;

(3) If $u_1 = u_2$ on C, then $u_1 \equiv u_2$ in G;

(4) If $\partial u_1/\partial n = \partial u_2/\partial n$ on C, then $u_1-u_2 =$ constant in G.

1065. Prove that if u is harmonic in the circle $|z| < R$ and continuous in the closed circle $|z| \leqslant R$, then

$$u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(Re^{i\theta})\,d\theta = \frac{1}{\pi R^2}\iint_{|z|<R} u(re^{i\phi})\,r\,dr\,d\phi\,.$$

HINT. It follows from the equality $\int_C \frac{\partial v}{\partial n}\,ds = 0$ that the integral $\int_{|z|=r} u\,d\phi$ does not depend on r; find its value by passing to the limit as $r \to 0$, and then pass to the limit as $r \to R$.

1066. Integrals of the form

$$\iint_G \varrho(\zeta)\log\frac{1}{|\zeta-z|}\,d\xi\,d\eta, \qquad \int_C \varrho(\zeta)\log\frac{1}{|\zeta-z|}\,|d\zeta|\,,$$

$$\int_C \nu(\zeta)\frac{\partial}{\partial n}\log\frac{1}{|\zeta-z|}\,|d\zeta|$$

($\zeta = \xi+i\eta$, n is the normal to C) are known respectively as the *logarithmic potential*, the *logarithmic potential of a simple layer* and the *logarithmic potential of a double layer.*

Verify the following equalities:

$$(1)\ \frac{1}{2\pi}\int_0^{2\pi}\log\frac{1}{|\zeta-z|}\,d\theta = \begin{cases}\log\dfrac{1}{|z|}, & \text{if } |z|>R\\[2ex] \log\dfrac{1}{R}, & \text{if } |z|\leqslant R\end{cases} \qquad (\zeta = Re^{i\theta});$$

$$(2)\ \frac{1}{\pi R^2}\iint_{|\zeta|<R}\log\frac{1}{|\zeta-z|}\,d\xi\,d\eta = \begin{cases}\log\dfrac{1}{|z|}, & \text{if } |z|>R,\\[2ex] \log\dfrac{1}{R}+\dfrac{1}{2}\left[1-\left(\dfrac{|z|}{R}\right)^2\right], & \text{if } |z|\leqslant R;\end{cases}$$

$$(3)\ \int_C \nu(\zeta)\frac{\partial}{\partial n}\log\frac{1}{|\zeta-z|}\,|d\zeta| = \int_C \nu(\zeta)\,d\arg(\zeta-z)$$

(in traversing C the normal n is taken on the left);

$$(4)\ \int_{-a}^{a}\frac{\partial}{\partial\eta}\log\frac{1}{|\zeta-z|}\,d\xi = \begin{cases}\phi, & \text{if } \operatorname{Im} z>0,\\ -\phi, & \text{if } \operatorname{Im} z<0,\end{cases}$$

where ϕ $(0 < \phi < \pi)$ is the angle subtended by the segment $(-a, a)$ at the point z $(a > 0)$.

Green's function $g(x, y, \xi, \eta)$ of the domain G (abbreviated to $g(z, \zeta)$, $z = x+iy$, $\zeta = \xi+i\eta$) is defined as a harmonic function of both pairs of variables x, y and ξ, η, which is equal to zero on the boundary of the domain G, and has a singularity at $z = \zeta$, where

$$g(z, \zeta) = \log \frac{1}{|z-\zeta|} + \text{(a harmonic function)}.$$

Green's function is symmetrical with respect to its arguments, that is, $g(z, \zeta) = g(\zeta, z)$ (see, for example, [1, Chapter VI, § 1]).

1067. Formulate the Dirichlet problem for harmonic functions which is equivalent to the finding of Green's function $g(z, \zeta)$.

1068. Let the function $w = f(z, \zeta)$ conformally map the simply connected Jordan domain G onto the disk $|w| < 1$, so that $f(\zeta, \zeta) = 0$ $(\zeta \in G)$.

Prove the relations

$$g(z, \zeta) = -\log |f(z, \zeta)|, \tag{1}$$

$$f(z, \zeta) = e^{-(g+ih)}, \tag{2}$$

where $h(z, \zeta)$ is the harmonic function conjugate to $g(z, \zeta)$.

1069. Using the relation (1) of problem **1068** find Green's function $g(z, \zeta)$ for the following domains:

(1) For the disk $|z| < R$;

(2) For the half-plane $\operatorname{Im} z > 0$;

(3) For the strip $0 < \operatorname{Im} z < 1$.

1070. Prove the following assertions (n is the inward normal, γ_r is the circle $|z-\zeta| = r$):

(1) If $u(z)$ is continuous near to $z = \zeta$, then

$$\lim_{r \to 0} \int_{\gamma_r} u(r) \frac{\partial g(z, \zeta)}{\partial n} ds = 2\pi u(z).$$

(2) If $u(z)$ is continuously differentiable near to $z = \zeta$, then

$$\lim_{r \to 0} \int_{\gamma_r} g(z, \zeta) \frac{\partial u(z)}{\partial n} ds = 0.$$

(3) If $u(z)$ is harmonic in G and continuously differentiable on C, then

$$u(\zeta) = \frac{1}{2\pi} \int_C u(z) \frac{\partial g(z, \zeta)}{\partial n} ds \qquad (z \in G).$$

HINT. In the formula

$$\int_C \left(u\frac{\partial g}{\partial n} - g\frac{\partial u}{\partial n}\right) ds = \int_{\gamma_r} \left(u\frac{\partial g}{\partial n} - g\frac{\partial u}{\partial n}\right) ds$$

pass to the limit as $r \to 0$.

1071. The *kernel functions of the domain*

$$K(z, \bar{\zeta}) = -\frac{2}{\pi}\frac{\partial^2 g(z, \zeta)}{\partial z \partial \bar{\zeta}}, \qquad L(z. \zeta) = -\frac{2}{\pi}\frac{\partial^2 g(z, \zeta)}{\partial z \partial \zeta}$$

are defined in terms of Green's function $g(z, \zeta)$.

Prove the following assertions:

(1) For the disk $|z| < R$

$$K(z, \bar{\zeta}) = \frac{R^2}{\pi(R^2 - z\bar{\zeta})^2}, \qquad L(z, \zeta) = \frac{1}{\pi(z-\zeta)^2}.$$

(2) $K(z, \bar{\zeta})$ is a function of $z, \bar{\zeta}$ analytic everywhere in D and $L(z, \zeta)$ is a function of z, ζ analytic everywhere in D except at the point $z = \zeta$, at which it has a pole of the second order,

$$L(z, \zeta) = \frac{1}{\pi(z-\zeta)^2} + l(z, \zeta),$$

where $l(z, \zeta)$ is a function of z, ζ which is analytic everywhere.

(3) $K(z, \bar{\zeta}) = \overline{K(\zeta, \bar{z})}$, $L(z, \zeta) = L(\zeta, z)$, $l(z, \zeta) = l(\zeta, z)$ (the symmetry of Green's function $g(z, \zeta) = g(\zeta, z)$ is considered known).

(4) If $f(z)$ is an arbitrary analytic function in G, continuous in the closed region $\bar{G}$, then

$$\left(f(z), \overline{K(z, \bar{\zeta})}\right) = \iint_G K(\zeta, \bar{z}) f(z)\, dx\, dy = f(\zeta)$$

(the "multiplicative property" of the kernel $K(z, \bar{\zeta})$).

(5) With the same conditions

$$\left(f(z), \overline{L(z, \zeta)}\right) = -\frac{2}{\pi}\iint_G \frac{\partial^2 g(z, \zeta)}{\partial \bar{z} \partial \bar{\zeta}} f(z)\, dx\, dy = 0$$

(the "orthogonality property" of the kernel $L(z, \zeta)$).

REMARK. See the literature indicated in the remark to problem **1054**.

§ 4. Poisson's integral, Schwarz's formula, harmonic measure

If the real function $u(\zeta) = u(R, \theta)$ is defined and piecewise continuous on the circumference $\zeta = Re^{i\theta}$ $(0 \leqslant \theta < 2\pi)$, then the *Poisson integral*

$$u(z) = u(r, \phi) = \frac{1}{2\pi} \int_0^{2\pi} u(R, \theta) \frac{R^2 - r^2}{R^2 - 2Rr \cos(\theta - \phi) + r^2} \, d\theta \qquad (1)$$

defines in the disk $|z| < R$ $(z = re^{i\phi})$ a harmonic function having at the points of continuity of $u(\zeta)$ boundary values equal to $u(\zeta)$:

$$\lim_{z \to \zeta} u(z) = u(\zeta)$$

$(z \to \zeta$ along any non-tangential path). The corresponding function $f(z) = u + iv$, analytic in the disk $|z| < R$, is defined by *Schwarz's formula*:

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} u(\zeta) \frac{\zeta + z}{\zeta - z} \, d\theta + iv(0)$$

($v(0)$ is an arbitrary real number).

1072. Prove the following assertions:

(1) $$\frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr \cos(\theta - \phi) + r^2} \, dr = 1;$$

(2) $$u(r, \phi) - u(R, \theta_0) = \frac{1}{2\pi} \int_0^{2\pi} [u(R, \theta) - u(R, \theta_0)] \frac{R^2 - r^2}{R^2 - 2Rr \cos(\theta - \phi) + r^2} \, d\theta;$$

(3) If $|u(R, \theta) - u(R, \theta_0)| < \varepsilon$ for $|\theta - \theta_0| < \alpha$, then

$$\frac{1}{2\pi} \int_{|\theta - \theta_0| < \alpha} |u(R, \theta) - u(R, \theta_0)| \frac{R^2 - r^2}{R^2 - 2Rr \cos(\theta - \phi) + r^2} \, d\theta < \varepsilon;$$

(4) If $|\theta - \theta_0| > \alpha$ and $|\phi - \theta_0| < \frac{\alpha}{2}$, then

$$R^2 - 2Rr \cos(\theta - \phi) + r^2 > 4Rr \sin^2 \frac{\alpha}{4};$$

(5) If $|\phi - \theta_0| < \frac{\alpha}{2}$ and the condition of part (3) is satisfied,

then

$$|u(r,\phi)-u(R,\theta_0)| < \varepsilon + \frac{M(R^2-r^2)}{2\pi A}$$

where

$$A = 4\pi Rr\sin^2\frac{\alpha}{4}, \qquad M = \int_0^{2\pi} |u(R,\theta)-u(R,\theta_0)|\,\mathrm{d}\theta\,.$$

1073. Prove that if $\zeta = Re^{i\theta}$, $z = re^{i\phi}$, then

$$\frac{\zeta+z}{\zeta-z} = \frac{(R^2-r^2)+i2Rr\sin(\phi-\theta)}{R^2-2Rr\cos(\theta-\phi)+r^2}\,.$$

Using this identity obtain the following expansions:

$$u(z) = u(0)+\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n (a_n\cos n\phi+b_n\sin n\phi)\,,$$

$$v(z) = v(0)+\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n (-b_n\cos n\phi+a_n\sin n\phi)\,,$$

$$f(z) = f(0)+\sum_{n=1}^{\infty} c_n z^n, \qquad f(0) = u(0)+iv(0)\,,$$

where

$$a_n = \frac{1}{\pi}\int_0^{2\pi} u(R,\theta)\cos n\theta\,\mathrm{d}\theta, \qquad b_n = \frac{1}{\pi}\int_0^{2\pi} u(R,\theta)\sin n\theta\,\mathrm{d}\theta\,,$$

$$c_n = \frac{a_n-ib_n}{R^n} = \frac{1}{\pi R^n}\int_0^{2\pi} u(R,\theta)e^{-in\theta}\,\mathrm{d}\theta\,.$$

1074. Prove that for a harmonic function $u(z)$ the Dirichlet integral is given by

$$D(u) = \iint_{|z|<R} (u_x^2+u_y^2)\,\mathrm{d}x\,\mathrm{d}y = \pi\sum_{n=1}^{\infty} n(a_n^2+b_n^2)$$

(the coefficients a_n and b_n are defined in the preceding problem). It is possible that both sides of the equation are equal to infinity.

HINT. Pass to polar coordinates (r, ϕ) and prove that for $r < R$

$$D_r(u) = \iint\limits_{|z|<r} (u_x^2+u_y^2)\mathrm{d}x\,\mathrm{d}y = \pi \sum_{n=1}^{\infty} n\left(\frac{r}{R}\right)^{2n}(a_n^2+b_n^2).$$

1075. Prove that for the continuous function

$$u(R, \theta) = \sum_{n=1}^{\infty} \frac{\sin n!\,\theta}{n^2}$$

the Dirichlet integral determines a function $u(z)$ harmonic in the circle $|z| < R$ with boundary values $u(R, \theta)$ and an infinite Dirichlet integral $D(u) = \infty$.

1076. By means of Poisson's integral solve the exterior Dirichlet problem for the circle $|z| < R$: find the function $u(z)$ harmonic in the domain $|z| > R$, regular at infinity and possessing given boundary values $u(\zeta)$ on the circle $|\zeta| = R$. Determine the value of $u(\infty)$.

HINT. Make the substitution $z_1 = \dfrac{R^2}{\bar{z}}$

1077. Prove that for $|z| > R$, Schwarz's formula (see the introduction to § 4) defines an analytic function $f_1(z) = u_1(z) + iv_1(z)$ regular at infinity and that the following expansions hold:

$$f_1(z) = -\overline{f\left(\frac{R^2}{\bar{z}}\right)} = -\overline{f(0)} - \sum_{n=1}^{\infty} \frac{c_{-n}}{z^n}, \qquad c_{-n} = R^{2n}\bar{c}_n,$$

$$u_1(z) = -\left\{u(0) + \sum_{n=1}^{\infty}\left(\frac{R}{r}\right)^n (a_n \cos n\phi + b_n \sin n\phi)\right\},$$

$$v_1(z) = -v(0) + \sum_{n=1}^{\infty}\left(\frac{R}{r}\right)^n (-b_n \cos n\phi + a_n \sin n\phi),$$

where a_n, b_n, c_n are defined just as in problem **1073**.
Here $\operatorname{Re} f_1(\zeta) = -\operatorname{Re} f(\zeta) = -u(\zeta)$, $\operatorname{Im} f_1(\zeta) = \operatorname{Im} f(\zeta)$.

HINT. In Schwarz's formula for $|z| > R$ make the substitution $z = \dfrac{R^2}{\bar{z}_1}$ and make use of the fact that $\zeta = R^2/\bar{\zeta}$ on the circumference $|\zeta| = R$.

In problems **1078–1083** find the functions $f(z)$ $(|z| < R)$ and $f_1(z)$ $(|z| > R)$ defined by Schwarz's formula if $u(\zeta)$ is the given function.

1078. (1) $u(\zeta) = \operatorname{Re}[\phi(\zeta)+\psi(\zeta)]$,
(2) $u(\zeta) = \operatorname{Im}[\phi(\zeta)+\psi(\zeta)]$,
where $\phi(z)$ is a function analytic for $|z| \leqslant R$, and $\psi(z)$ is a function analytic for $|z| \geqslant R$.

1079. $u(\zeta) = \operatorname{Re} \zeta^n$.

1080. $u(\zeta) = \operatorname{Re} 1/\zeta^n$.

1081. $u(\zeta) = \operatorname{Re} \log \dfrac{\zeta}{\zeta-1}$ $(R > 1)$.

1082. $u(\zeta) = \operatorname{Re} \sqrt{\left(\dfrac{\zeta}{\zeta-1}\right)}$ $\left(\text{for } \zeta > 1, \sqrt{\left(\dfrac{\zeta}{\zeta-1}\right)} > 0,\ R > 1\right)$.

1083. $u(\zeta) = \operatorname{Re} \log \zeta$.

1084. Prove that Schwarz's formula can be written in the form

$$f(z) = \frac{1}{\pi i} \int_{|\zeta|=R} \frac{u(\zeta)\,d\zeta}{\zeta-z} - \overline{f(0)}.$$

HINT. Use the equality

$$\frac{\zeta+z}{\zeta(\zeta-z)} = \frac{2}{\zeta-z} - \frac{1}{\zeta}.$$

1085. Obtain formulae similar to Poisson's integral and Schwarz's formula for the upper half-plane $\operatorname{Im} z > 0$, that is, express the harmonic function $u(z)$ and the analytic function $f(z) = u(z)+iv(z)$ in terms of $u(t)$ $(-\infty < t < \infty)$.

HINT. Make use of the conformal mapping of the half-plane onto the circle.

1086. Derive Schwarz's formula for the strip $0 < \operatorname{Im} z < 1$.

HINT. Use the conformal mapping of the strip onto the half-plane.

The *harmonic measure* $\omega(z, \alpha, G)$ of the boundary arc α at the point z with respect to the domain G is defined as a bounded function, harmonic in G, which is equal to 1 at the interior points of the arc α and to 0 at the interior points of the remaining part of the boundary. The harmonic measure $\omega(z, \alpha, G)$ is an invariant in conformal mappings.

In problems **1087–1090** the domain G is the circle $|z| < 1$ and $\omega(z, \theta_1, \theta_2)$ is the harmonic measure of the arc $\alpha = (\theta_1, \theta_2)$, $w = e^{i\theta}$, $\theta_1 \leqslant \theta \leqslant \theta_2$.

1087. Using Poisson's integral prove that

$$\omega(z, \theta_1, \theta_2) = \frac{1}{2\pi} \int_{\theta_1}^{\theta_2} \frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2} d\theta,$$

and in particular, that $\omega(0, \theta_1, \theta_2) = (\theta_2-\theta_1)/2\pi$.

1088. Find the level lines of the function $d\omega(z, \theta_1, \theta)/d\theta$ for a fixed value of θ ($z = re^{i\phi}$ is a variable point).

HINT. Prove that

$$\frac{d\omega}{d\theta} = \frac{1}{2\pi} \left| \frac{w'-z}{w-z} \right|,$$

where w' is the end of the chord from w which passes through z.

1089. Let us denote by w' the end of the chord from w which passes through the point z. Let α be the arc (θ_1, θ_2), and $\alpha'(z)$ the arc described by the point w', when w traverses the arc α. Prove that the length of the arc $\alpha'(z)$ equals $2\pi\omega(z, \theta_1, \theta_2)$.

1090. Find the level lines of the harmonic measure $\omega(z, \theta_1, \theta_2)$ of the arc (θ_1, θ_2). Using this prove that the integral defining the harmonic measure (see problem **1087**), actually has the limiting values 1 on (θ_1, θ_2) and 0 on the complement (interior points of the arc arc considered).

1091. For the half-plane $\operatorname{Im} z > 0$ determine the harmonic measure $\omega(z, a, b)$ of the segment (a, b), of the ray $(-\infty, b)$ and of the ray (a, ∞). What is the geometrical meaning of these harmonic measures?

1092. Find the harmonic measures of the boundary rays of the angular sector.

$$0 < \arg z < \gamma.$$

1093. For the semicircle $|z| < R$, $\operatorname{Im} z > 0$ find the harmonic measures of the diameter Δ and of the semicircular arc Γ, and also the level lines of these harmonic measures.

1094. Find the harmonic measure of the bounding semicircular arc Γ of the domain $|z| > R$, $\operatorname{Im} z > 0$.

1095. Find the harmonic measure of the boundary circular arc Γ of the domain $|z| > R$, $0 < \arg z < 2\pi$.

1096. Find the harmonic measure of the boundary circumferences of the annulus $r < |z| < R$.

In problems **1097–1101** the domain G is bounded by a compound contour Γ, consisting of n simple, smooth contours $\Gamma_\nu (\nu = 1, 2, \dots, n)$. The contour is traversed in the positive direction with respect to the domain G; the normal n is inward with respect to the domain G. The *period* with respect to Γ_ν of a function analytic in G is defined as the integral

$$\int_{\Gamma_\nu} df(z)$$

(see page 16).

1097†. Prove that if the harmonic function $u(z)$ is single valued in G, then the period of the analytic function $f(z) = u(z) + iv(z)$ along Γ_ν is equal to

$$-\int_{\Gamma_\nu} \frac{\partial u}{\partial n} ds .$$

1098. Prove that for the complex Green function $g + ih$ of the domain G ($g(z, \zeta)$ is the Green function of the domain G, $h(z, \zeta)$ its harmonic conjugate) the period along Γ_ν equals $2\pi\omega_\nu(z)$, where $\omega_\nu(z)$ is the harmonic measure of Γ_ν with respect to the domain G. Prove that

$$\sum_{\nu=1}^{n} \omega_\nu(\zeta) \equiv 1 .$$

HINT. The function $u(\zeta)$ harmonic in G can be represented in the form

$$u(\zeta) = \frac{1}{2\pi} \int_{\Gamma} u(z) \frac{\partial g(z, \zeta)}{\partial n} ds$$

(see problem **1070**).

1099. Express in terms of $\omega_\nu(z)$ the function $u(z)$ which is bounded, harmonic in G and has constant values c_ν on the $\Gamma_\nu (\nu = 1, 2, \dots, n)$.

1100. Prove that for a function $v(z)$ which is conjugate to a function $u(z)$, single valued and harmonic in $\overline{G}$, the periods p_ν along Γ_ν can be represented in the form

$$p_\nu = -\int_{\Gamma} u(z) \frac{\partial \omega_\nu(z)}{\partial n} ds .$$

† For problems **1097–1104** see the references indicated in the remarks to problem **1054**.

1101. Let $\tilde{\omega}_\nu(z)$ be the harmonic function conjugate to $\omega_\nu(z)$ and $p_{\mu\nu}$ the periods of the function $w_\nu(z) = \omega_\nu(z) + i\tilde{\omega}_\nu(z)$ along Γ_μ.

(1) Prove that

$$p_{\mu\nu} = p_{\nu\mu} \qquad (\mu, \nu = 1, 2, \dots, n).$$

HINT. Use the representation

$$p_{\mu\nu} = -\frac{1}{2\pi} \int_\Gamma \omega_\mu \frac{\partial \omega_\nu}{\partial n} ds.$$

(2) Prove that

$$\sum_{\nu=1}^{n} p_{\mu\nu} = 0 \qquad (\mu = 1, 2, \dots, n).$$

1102. Let $c_\nu (\nu = 1, 2, \dots, n)$ be arbitrary real numbers. Prove that if $p_{\nu\mu}$ are the numbers defined in problem **1101**, then the quadratic form

$$\sum_{\nu,\mu}^{n} p_{\nu\mu} c_\nu c_\mu \geqslant 0,$$

the case of equality occurring only when allthe c_ν are equal.

HINT. Apply the formula

$$D(\omega) = -\int_\Gamma \omega \frac{\partial \omega}{\partial n} ds$$

to the harmonic function $\omega(z) = \sum_{\nu=1}^{n} c_\nu \omega_\nu(z)$ (see problem **1064**; the sign is reversed because n is now the inward normal).

1103. Prove that the quadratic form

$$\sum_{\nu,\mu=1}^{n-1} p_{\nu\mu} c_\nu c_\mu$$

is positive definite, that is, positive for all sets of values $\{c_\nu\}$, excluding $c_1 = c_2 = \dots = c_{n-1} = 0$.

1104. Prove that the system of equations

$$\sum_{\nu=1}^{n-1} p_{\nu\mu} A_\mu = B_\mu \qquad (\mu = 1, 2, \dots, n-1)$$

(the A_μ are unknowns) has a unique solution for any B_μ. Using this, prove that for any function $u(z)$ harmonic in $\overline{G}$, generally speaking not single-valued, it is possible to select constants $A_1, A_2, \ldots, A_{n-1}$ in such a way that the harmonic function

$$u_1(z) = u(z) + \sum_{\nu=1}^{n-1} A_\nu \omega_\nu(z)$$

is single-valued in G.

§ 5. Some singular integrals†

Let $h(\phi)$ be an integrable function with period 2π; a_0, a_n, $b_n (n = 1, 2, \ldots)$ its Fourier coefficients; $f(z) = u(z) + iv(z)$ and $f_1(z) = u_1(z) + iv_1(z)$ functions analytic respectively in the disk $|z| < 1$ and in the domain $|z| > 1$, defined by means of Schwarz's integral

$$\frac{1}{2\pi} \int_0^{2\pi} h(t) \frac{e^{it} + z}{e^{it} - z} dt \qquad (z = re^{i\phi}). \tag{1}$$

The function $\tilde{h}(\phi)$, defined by means of a singular integral with the Hilbert kernel $\frac{1}{2\pi} \cot \frac{\phi - t}{2}$:

$$\tilde{h}(\phi) = T[h(\phi)] = \frac{1}{2\pi} \int_0^{2\pi} h(t) \cot \frac{\phi - t}{2} dt = \frac{1}{2\pi} \lim_{\varepsilon \to 0} \left(\int_0^{\phi - \varepsilon} + \int_{\phi + \varepsilon}^{2\pi} \right), \tag{2}$$

is said to be conjugate to the function $h(\phi)$. The series

$$\sum_{n=1}^{\infty} (-b_n \cos n\phi + a_n \sin n\phi) \tag{3}$$

is said to be conjugate to the Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\phi + b_n \sin n\phi) \tag{4}$$

of the function $h(\phi)$.

1105. Prove the following assertions:

(1) The principal value of Schwarz's integral (1) at the point $e^{i\phi}$, if it exists, is equal to $i\tilde{h}(\phi)$.

(2) If $h(\phi) = \text{const}$, then $\tilde{h}(\phi) \equiv 0$.

† Problems **1119–1128** and **1142–1143** on the approximate evaluation of singular integrals were composed by B. A. Vertgeim.

(3) If $h(\phi) \in \operatorname{Lip}\alpha\ (0 < \alpha \leqslant 1)$ (see page 152), then $\tilde{h}(\phi)$ exists and is continuous. In addition, if $\alpha < 1$, then $\tilde{h}(\phi) \in \operatorname{Lip}\alpha$.

HINT. Write $\tilde{h}(\phi)$ in the form

$$\tilde{h}(\phi) = \frac{1}{2\pi}\int_0^{2\pi} [h(t)-h(\phi)]\cot\frac{\phi-t}{2}\,dt = -\frac{1}{2\pi}\int_0^{\pi} [h(\phi+t)-h(\phi-t)]\cot\frac{t}{2}\,dt.$$

1106. Let $h(\phi)$ be continuous and have a piecewise continuous derivative $h'(\phi)$. Prove the following assertions:

(1) $\tilde{h}(\phi) = \lim\limits_{r\to 1} f(re^{i\phi}) = \lim\limits_{r\to 1} f_1(re^{i\phi})$ (the definition of the functions $f(z)$ and $f_1(z)$ are given at the beginning of the section).

HINT. Show that

$$\operatorname{Im} f(re^{i\phi}) = \frac{1}{2\pi}\int_0^{2\pi} [h(t)-h(\phi)]\frac{2r\sin(\phi-t)}{1-2r\cos(\phi-t)+r^2}\,dt.$$

(2) The function $f(z) = \dfrac{a_0}{2} + \sum\limits_{n=1}^{\infty} c_n z^n$ $(c_n = a_n - ib_n)$ is continuous in the closed disk $|z| \leqslant 1$ and $f(e^{i\phi}) = h(\phi) + i\tilde{h}(\phi)$.

HINT. With the conditions of the problem $a_n = O\left(\dfrac{1}{n^2}\right)$, $b_n = O\left(\dfrac{1}{n^2}\right)$.

(3) The function $f_1(z) = -\sum\limits_{n=1}^{\infty} \dfrac{\overline{c_n}}{z^n}$ is continuous in the domain $|z| \geqslant 1$ and $f_1(e^{i\phi}) = -h(\phi) + i\tilde{h}(\phi)$.

(4) $\tilde{h}(\phi) = \sum\limits_{n=1}^{\infty} (-b_n \cos n\phi + a_n \sin n\phi)$ and the series converges uniformly.

(5) $\int_0^{2\pi} \tilde{h}(\phi)\,d\phi = 0$.

1107. Let $\psi(z)$ be an analytic function in the disk $|z| \leqslant 1$ and $\psi_1(z)$ an analytic function in the domain $|z| \geqslant 1$. Prove the following properties of the transformation (2):

(1) $T[\operatorname{Re}\psi(e^{i\phi})] = \operatorname{Im}\psi(e^{i\phi}) - \operatorname{Im}\psi(0)$;

(2) $T[\operatorname{Im}\psi(e^{i\phi})] = -\operatorname{Re}\psi(e^{i\phi}) + \operatorname{Re}\psi(0)$;

(3) $T[\operatorname{Re}\psi_1(e^{i\phi})] = -\operatorname{Im}\psi_1(e^{i\phi}) + \operatorname{Im}\psi_1(\infty)$;

(4) $T[\operatorname{Im}\psi_1(e^{i\phi})] = \operatorname{Re}\psi_1(e^{i\phi}) - \operatorname{Re}\psi_1(\infty)$.

In problems **1108–1116** find the function $\tilde{h}(\phi)$ from the given function $h(\phi)$.

1108. $\cos n\phi$.

1109. $\sin n\phi$.

1110. $\dfrac{1-a\cos\phi}{1-2a\cos\phi+a^2}$ $\quad(-1<a<1)$.

1111. $\log(1-2a\cos\phi+a^2)$ $(-1<a<1)$.

1112. $\cos^{2m}\phi$ (m is a natural number).

1113. $\dfrac{\sin(n+\frac{1}{2})\phi}{2\sin\frac{1}{2}\phi}$ (n is a natural number).

1114. $|\phi|$ $(-\pi\leqslant\phi\leqslant\pi)$.

1115. ϕ $(-\pi\leqslant\phi\leqslant\pi)$.

1116. ϕ $(0\leqslant\phi\leqslant 2\pi)$.

1117. Prove the following assertions:

(1) If $h(\phi)$ is an even (odd) function, then $\tilde{h}(\phi)$ is an odd (even) function.

(2) If $h(\phi+\omega)=h(\phi)$, then $\tilde{h}(\phi+\omega)=\tilde{h}(\phi)$.

(3) If $h(\phi+\omega)=-h(\phi)$, then $\tilde{h}(\phi+\omega)=-\tilde{h}(\phi)$.

1118. Let $F(t,\phi)$, $h(\phi)$ and $h_1(\phi)$ be periodic functions ($F(t,\phi)$ with respect to both variables) with period 2π, belonging to the class Lip α $(0<\alpha\leqslant 1)$.

Prove the following assertions:

$$(1)\quad \int_0^{2\pi}\mathrm{d}\phi\int_0^{2\pi}F(t,\phi)\cot\frac{\phi-t}{2}\,\mathrm{d}t=\int_0^{2\pi}\mathrm{d}t\int_0^{2\pi}F(t,\phi)\cot\frac{\phi-t}{2}\,\mathrm{d}\phi$$

$$=2\pi^2\sum_{n=1}^{\infty}(A_n-B_n),$$

where

$$A_n=\frac{1}{\pi^2}\int_0^{2\pi}\int_0^{2\pi}F\cos nt\sin n\phi\,\mathrm{d}t\,\mathrm{d}\phi,$$

$$B_n=\frac{1}{\pi^2}\int_0^{2\pi}\int_0^{2\pi}F\sin nt\cos n\phi\,\mathrm{d}t\,\mathrm{d}\phi;$$

$$(2)\quad \int_0^{2\pi}\left\{\frac{1}{2\pi}\int_0^{2\pi} h(t)\cot\frac{\phi-t}{2}\,dt\right\}h_1(\phi)\,d\phi$$

$$=\int_0^{2\pi}\left\{\frac{1}{2\pi}\int_0^{2\pi} h_1(\phi)\cot\frac{\phi-t}{2}\,d\phi\right\}h(t)\,dt=\pi\sum_{n=1}^{\infty}(a_nb_n^{(1)}-b_na_n^{(1)}),$$

where (a_n, b_n) and $(a_n^{(1)}, b_n^{(1)})$ are the Fourier coefficients of the functions $h(\phi)$ and $h_1(\phi)$;

$$(3)\quad \frac{1}{2\pi}\int_0^{2\pi}\left\{\frac{1}{2\pi}\int_0^{2\pi} h(t)\cot\frac{\tau-t}{2}\,dt\right\}\cot\frac{\phi-\tau}{2}\,d\tau$$

$$=-h(\phi)+\frac{1}{2\pi}\int_0^{2\pi} h(\phi)\,d\phi$$

or, if $\tilde{h}=T(h)$,

$$T[T(h)]=T(\tilde{h})=-h(\phi)+\frac{1}{2\pi}\int_0^{2\pi} h(\phi)\,d\phi\,.$$

1119. Applying the rectangle formula for the approximate evaluation of the singular integral

$$\tilde{h}(\phi)=-\frac{1}{2\pi}\int_0^{\pi}[h(\phi+t)-h(\phi-t)]\cot\frac{t}{2}\,dt,$$

obtain the formula

$$\tilde{h}_{k,n}=-\frac{1}{2n}\sum_{i=1}^{n}(h_{k+i,\,n}-h_{k-i,\,n})\cot(2i-1)\frac{\pi}{4n}, \qquad (5)$$

where

$$h_{k,n}=h\left(\frac{\pi k}{2n}\right),\ \tilde{h}_{k,n}=\tilde{h}\left(\frac{2\pi k}{n}\right)\quad (k=0,\ \pm1,\ \pm2,\ \dots).$$

In particular, if $n=5$ and

$$h_k=h\left(\frac{\pi k}{10}\right),\quad \tilde{h}_k=\tilde{h}\left(\frac{\pi k}{10}\right),$$

then

$$\tilde{h}_k = -\frac{1}{10}\Big[(h_{k+1}-h_{k-1})\cot\frac{\pi}{20}+(h_{k+3}-h_{k-3})\cot\frac{3\pi}{20}$$
$$+(h_{k+5}-h_{k-5})\cot\frac{5\pi}{20}+(h_{k+7}-h_{k-7})\cot\frac{7\pi}{20}$$
$$+(h_{k+9}-h_{k-9})\cot\frac{9\pi}{20}\Big]. \qquad (6)$$

1120. Let $\Phi(t) = -\frac{1}{2\pi}[h(\phi+t)-h(\phi-t)]\cot\frac{t}{2}$.

Assuming that $h(\phi)$ can be continuously differentiated three times, prove that

$$\max_{0\leqslant t\leqslant\pi}|\Phi''(t)| \leqslant \frac{1}{\pi}\left[M_1+\left(\frac{\pi^2}{2}+\frac{8}{3}\right)M_3\right]$$

where

$$M_k = \max_{0\leqslant\phi\leqslant\pi}|h^{(k)}(\phi)| \qquad (k=1,3).$$

Using this and the estimate of error R_n of the rectangle formula for the calculation of $\tilde{h}(\phi)$:

$$R_n = \frac{(b-a)^3}{24n^2}\Phi''(\xi), \qquad a\leqslant\xi\leqslant b \qquad (a=0, b=\pi),$$

obtain the estimate

$$|R_n| \leqslant \frac{\pi^2}{24n^2}\left[M_1+\left(\frac{\pi^2}{2}+\frac{8}{3}\right)M_3\right]. \qquad (7)$$

In particular, if $n=5$

$$|R| < 1\cdot 65\times 10^{-2}(M_1+7\cdot 61M_3). \qquad (8)$$

HINT. Expand $\Phi(t)$ by Taylor's formula with remainder term in Lagrange's form (with h''') and use the inequalities

$$\frac{\alpha}{\sin\alpha}\leqslant\frac{\pi}{2}, \quad \alpha\cot\alpha\leqslant 1, \quad \frac{1-\alpha\cot\alpha}{\sin^2\alpha}\leqslant 1$$

and

$$\frac{\alpha^3\cos\alpha}{\sin^3\alpha}\leqslant 1 \text{ for } 0\leqslant\alpha\leqslant\frac{\pi}{2}\left(\alpha=\frac{t}{2}\right).$$

1121. Prove that if $h(\phi+\pi)=h(\phi)$, then formula (6) (see problem **1119**) acquires the form

$$\tilde{h}_k=-\frac{1}{10}\left[(h_{k+1}-h_{k-1})2\cot\frac{\pi}{10}+(h_{k+3}-h_{k-3})\cot\frac{3\pi}{10}\right], \quad (9)$$

and if $h(\phi+\pi)=-h(\phi)$, then

$$\tilde{h}_k=-\frac{1}{10}\left[(h_{k+1}-h_{k-1})\frac{2}{\sin\frac{\pi}{10}}+(h_{k+3}-h_{k-3})\frac{2}{\sin\frac{3\pi}{10}}+2h_{k+5}\right]. \quad (10)$$

In problems **1122–1128** calculate the values of $\tilde{h}_k$ to five significant figures by one of the formulae (6), (9) or (10) and compare with the exact values. Also estimate the errors by formula (8) and compare with the actual errors.

1122. $h(\phi)=\cos 2\phi$.
1123. $h(\phi)=\cos 3\phi$.
1124. $h(\phi)=\cos 4\phi$.
1125. $h(\phi)=\sin\phi$.
1126. $h(\phi)=\sin 3\phi$.

1127. $h(\phi)=\phi\left(1-\frac{\phi}{\pi}\right)$, $0\leqslant\phi\leqslant\pi$ and $h(-\phi)=-h(\phi)$.

1128. $h(\phi)=\begin{cases}\phi, & \text{if } 0\leqslant\phi\leqslant\pi/2,\\ \pi-\phi, & \text{if } \pi/2\leqslant\phi\leqslant\pi,\end{cases}$ $h(-\phi)=-h(\phi)$.

REMARK. In problems **1127**, **1128** compare the result with calculations, based on the expansion of $h(\phi)$ in a Fourier series.

In problems **1129**, **1130** singular integrals are considered of the form

$$\tilde{h}(x)=T[h(x)]=-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{h(t)\,dt}{t-x}, \quad (11)$$

connected with Schwarz's integral for the half-plane (see problem **1085**)

$$\frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{h(t)\,dt}{t-z}. \quad (12)$$

If the integral in the usual sense diverges, its principal value must be considered. The function $\tilde{h} = T(h)$ is said to be conjugate to the function h.

1129. Prove the following assertions:

(1) The principal value of the Schwarz integral (12) at the point x, if it exists, is equal to $i\tilde{h}(x)$.

(2) If $h(x) = \text{const}$, then $\tilde{h}(x) \equiv 0$, if the principal value of the integral (11) is considered not only at the point x, but also at ∞;

$$\int_{-\infty}^{\infty} \frac{dt}{t-x} = \lim_{\substack{R\to\infty \\ \varepsilon\to 0}} \left(\int_{-R}^{x-\varepsilon} + \int_{x+\varepsilon}^{R} \right) = 0.$$

(3) If $h(x) \in \mathrm{Lip}\,\alpha$ $(0 < \alpha \leqslant 1)$ on any finite interval and the integral (7) converges on $\pm\infty$ (e.g., if $|h(x)| = O\left(\frac{1}{|x|^{\beta}}\right)$, $\beta > 0$, for large $|x|$), then $\tilde{h}(x)$ exists and can be expressed by the formulae

$$\tilde{h}(x) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{h(t)-h(x)}{t-x}\, dt = -\frac{1}{\pi} \int_{0}^{\infty} \frac{h(x+t)-h(x-t)}{t}\, dt. \quad (13)$$

(4) Prove that with the conditions of section (3) $\tilde{h}(x)$ is continuous and in addition, if $\alpha < 1$, $\tilde{h}(x) \in \mathrm{Lip}\,\alpha$.

(5) If $h(-x) = \pm h(x)$, then $\tilde{h}(-x) = \mp \tilde{h}(x)$.

(6) If $h(x+\omega) = \pm h(x)$, then $\tilde{h}(x+\omega) = \mp \tilde{h}(x)$.

1130. Let $f(z)$ and $f_1(z)$ be analytic functions defined by the Schwarz integral (12) in the half-planes $\operatorname{Im} z > 0$ and $\operatorname{Im} z < 0$ respectively. Prove that if $h(x) \in \mathrm{Lip}\,\alpha$ $(0 < \alpha \leqslant 1)$ uniformly on the x-axis and $|h(x)| = O\left(\frac{1}{|x|^{\beta}}\right)$, $\beta > 0$, at ∞, then

(1) $f_1(z) = -\overline{f(\bar{z})}$ $(\operatorname{Im} z < 0)$;

(2) $f(x) = \lim_{y\to 0} f(x+iy) = h(x)+i\tilde{h}(x)$ $(y > 0)$,

$$f_1(x) = \lim_{y\to 0} f_1(x+iy) = -h(x)+i\tilde{h}(x) \quad (y < 0).$$

HINT. In order to estimate the difference $\operatorname{Re} f(x+iy)-h(x)$ as $y \to 0$ use the fact that

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{y\,dt}{(t-x)^2+y^2}=1,$$

and divide the integral $\int_{-\infty}^{\infty} = \operatorname{Re} f(x+iy)-h(x)$ into the integrals $\int_{-\infty}^{x-1}+\int_{x-1}^{x+1}+\int_{x+1}^{\infty}$. To estimate the integral $\int_{0}^{\infty} = \operatorname{Im} f(x+iy)-\tilde{h}(x)$ (see (13)) subdivide it into the integrals $\int_{0}^{1}+\int_{1}^{\infty}$ and in the first of these make the substitution $t = uy$.

REMARK. The assertion of the problem can be significantly strengthened.

1131. (1) Let $\phi(z)$ and $\psi(z)$ be analytic functions in the half-planes $\operatorname{Im} z > 0$ and $\operatorname{Im} z < 0$ respectively, bounded at ∞ and continuous in the closed half-planes. Prove that if $\phi(x)$ and $\psi(x) \in \operatorname{Lip}\alpha\ (0 < \alpha \leqslant 1)$ and there exist transformations $T[\phi(x)]$, $T[\psi(x)]$ $\left(T(h_1+ih_2) = T(h_1)+iT(h_2)\right)$ and the limits

$$\lim_{R\to\infty}\frac{1}{\pi}\int_{0}^{\pi}\phi(Re^{i\theta})\,d\theta=\phi_\infty,\quad \lim_{R\to\infty}\frac{1}{\pi}\int_{0}^{\pi}\psi(Re^{-i\theta})\,d\theta=\psi_\infty,$$

then

$$T[\phi(x)] = -i[\phi(x)-\phi_\infty],\qquad T[\psi(x)] = i[\psi(x)-\psi_\infty].$$

HINT. Let $\phi(z) = u(z)+iv(z)$ and $\phi_\infty = u_\infty+iv_\infty$. Applying the results of problems **1085** and **1129**, we have $(z = x+iy)$

$$v(x) = \lim_{y\to 0}[\phi(z)-u(x)] = \lim_{y\to 0}\left\{-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{u(t)-u(x)}{t-z}\,dt\right\}+v_\infty,$$

$$T[u(x)] = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{u(t)-u(x)}{t-x}\,dt.$$

Estimating the difference of integrals it is possible to obtain

$$T[u(x)] = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{u(t)\,dt}{t-x} = v(x)-v_\infty,$$

which reduces to the first proved relation. The remaining one is proved similarly.

(2) In the conditions of part (1) let $\phi(z)$ and $\psi(z)$ have respectively the isolated singularities a ($\operatorname{Im} a > 0$) and b ($\operatorname{Im} b < 0$). Find $T[\phi(x)]$ and $T[\psi(x)]$.

In problems **1132–1137** find $\tilde{h}(x)$, $f(z)$ and $f_1(z)$ for the following functions $h(x)$ (the parameters λ, a, b are real numbers, $\lambda > 0$, $b > 0$, n is a natural number):

1132. $\cos \lambda x$.

1133. $\sin \lambda x$.

1134. $\dfrac{x}{x^2+b^2}$.

1135. $\dfrac{1}{1+x^4}$.

1136. $\operatorname{Re} 1/[x-(a+ib)]^n$.

1137. $\operatorname{Re} 1/[x-(a-ib)]^n$.

1138. Prove that

$$-\frac{1}{\pi}\int_0^\infty \frac{dt}{t^\lambda(t-x)} = \begin{cases} -\dfrac{\cot \lambda\pi}{x^\lambda}, & \text{if } x>0 \\ -\dfrac{1}{|x|^\lambda \sin \lambda\pi}, & \text{if } x<0 \end{cases} \qquad (0<\lambda<1).$$

HINT. Consider the integral of $1/[\zeta^\lambda(\zeta-x)]$ along the boundary of the annulus $r < |\zeta| < R$ (deleting the circle $|\zeta-x| < \varrho$, if $x > 0$) with a cut along the radial segment $r < \operatorname{Re}\zeta = t < R$, $\operatorname{Im}\zeta = 0$ (or the remaining part of it if $x > 0$) and pass to the limits as $r \to 0$, $R \to \infty$ (and $\varrho \to 0$, if $x > 0$). On the cut $\zeta^\lambda = t^\lambda$ above and $\zeta^\lambda = t^\lambda e^{2\pi i\lambda}$ below.

1139. Prove that

$$-\frac{1}{\pi}\int_a^\infty \frac{dt}{t^n(t-x)} = \frac{1}{\pi x^n}\log\left|\frac{x}{a}-1\right| + \frac{1}{\pi a^n}\sum_{k=1}^{n-1}\frac{1}{n-k}\left(\frac{a}{x}\right)^k$$

($x \neq a$, $a > 0$, n is a natural number). For $|x| < a$ the integral can be represented in the form $-\dfrac{1}{\pi a^n}\displaystyle\sum_{k=0}^{\infty}\frac{1}{n+k}\left(\frac{x}{a}\right)^k$. Both expansions are also valid for $x = -a$.

HINT. Use the expansion

$$\frac{1}{t^n(t-x)} = -\frac{1}{x^n}\left(\frac{1}{t}+\frac{x}{t^2}+\ldots+\frac{x^{n-1}}{t^n}\right)+\frac{1}{x^n(t-x)}.$$

1140. Prove that

$$-\frac{1}{\pi}\int_a^\infty \frac{dt}{t^\alpha(t-x)} = \begin{cases} -\dfrac{\cot \alpha\pi}{x^\alpha} - \dfrac{1}{\pi a^\alpha}\displaystyle\sum_{k=1}^{\infty} \frac{1}{k-\alpha}\left(\frac{a}{x}\right)^k, & \text{if } x > a, \\ -\dfrac{1}{|x|^\alpha \sin \alpha\pi} - \dfrac{1}{\pi a^\alpha}\displaystyle\sum_{k=1}^{\infty} \frac{1}{k-\alpha}\left(\frac{a}{x}\right)^k, & \text{if } x < -a, \\ -\dfrac{1}{\pi a^\alpha}\displaystyle\sum_{k=0}^{\infty} \frac{1}{k+\alpha}\left(\frac{x}{a}\right)^k, & \text{if } |x| < a \end{cases}$$

($a > 0$, $\alpha > 0$ are not integers). Both the latter expansions are valid also for $x = -a$, where the function $\tilde{h}(x)$ defined by the integral standing on the left remains continuous.

HINT. For $|x| > a$ consider the integral of $1/[\zeta^\alpha(\zeta - x)]$ along the boundary of the ring $a < |\zeta| < R$ with a cut along the radial segment $a < \operatorname{Re}\zeta = t < R$, $\operatorname{Im}\zeta = 0$ (and deleting the circle $|\zeta - x| < \varrho$, if $x > a$), and pass to the limit as $R \to \infty$ (and $\varrho \to 0$, if $x > a$). For $|x| < a$ expand the integrand in powers of x/t. The latter assertion is established by means of Abel's theorem on power series (for the latter expansion) and by means of the expansion of $1/\sin \alpha\pi$ in partial fractions (see problem **996**).

REMARK. If $0 < \alpha < 1$, thon for $|x| > a$ it is possiblc to use the integral of problem **1138**, from which it is necessary to subtract the integral $-\frac{1}{\pi}\int_0^a$, evaluated by expanding the integrand into a series of powers of t/x. The resulting expression retains its meaning (analyticity) also for $\alpha \geqslant 1$, and hence by analytic continuation, it is valid also for all $\alpha > 0$.

1141. Prove that if $h(x) = 0$ outside the interval $(-a, a)$, then for $|x| > a$

$$\tilde{h}(x) = -\frac{1}{\pi}\int_{-a}^{a} \frac{h(t)\,dt}{t-x} = \sum_{k=1}^{n} \frac{A_k}{x} + R_n(x),$$

where

$$A_k = \frac{1}{\pi}\int_{-a}^{a} t^{k-1}h(t)\,dt, \qquad |R_n| < \frac{1}{|x|^n(|x|-a)}\,\frac{1}{\pi}\int_{-a}^{a} |t|^n |h(x)|\,dt.$$

Find $\tilde{h}(x)$ in explicit form, if $h(x) = 0$ outside the interval $(-1,1)$ and in the interval $(-1,1)$ it equals respectively 1, $1-|x|$, sign x, and compare the result with the expansion in powers of $1/x$ given above.

For an approximate calculation of $\tilde{h}(x)$ for small values of x, $|x| < a$, the integrals along the infinite intervals $(-\infty, -b)$, (b, ∞) and along the interval $(-b, b)$, where $b > a$ are considered separately. For the interval $(-b, b)$ the formula

$$-\frac{1}{\pi}\int_{-b}^{b}\frac{h(t)\,dt}{t-x_{2k}} = -\frac{2}{\pi}\sum_{i=-n}^{n-1}\frac{h_{2i+1}}{2i-2k+1}+R_{2k}$$

can be obtained from the rectangle formula (here $x_k = k\frac{b}{2n}$; $k = 0, \pm 1, \pm 2, \ldots, \pm 2n$; $h(x_k) = h_k$; R_{2k} is the remainder). A similar formula is obtained for the points x_{2k+1}. To calculate $\tilde{h}_k = \tilde{h}(x_k)$ we take $|x_k| \leqslant a$. The integral along the infinite interval (b, ∞) is evaluated by means of the series expansion

$$-\frac{1}{\pi}\int_{b}^{\infty}\frac{h(t)\,dt}{t-x} = \sum_{k=0}^{\infty}c_k x^k$$

where

$$c_k = -\frac{1}{\pi}\int_{b}^{\infty}t^{-k-1}\,h(t)\,dt.$$

The integral along the infinite interval $(-\infty, -b)$ is treated similarly.

1142. Evaluate $\tilde{h}(x)$ for $h(x) = e^{-x^2}$ at the points $x = 1, 2, 3, 4, 5$. Take $a = 5$, $b = 6$, $n = 15$. The calculation is to be taken to five significant figures. Show that the integrals along the infinite intervals can be neglected. For comparison evaluate $\tilde{h}(x)$ at the points $x = 2, 3, 4, 5$, by the asymptotic expansion given in problem **946** (the terms of the series are to be used as long as they decrease).

1143. Evaluate $\tilde{h}(x)$ for $h(x) = 1/(1+x^2)$ at the points $x = 1$, $x = 5$, taking $b = 10$, $n = 25$. Compare the results with the exact values.

In problems **1144–1147** singular integrals are considered of the form

$$\tilde{h}(\tau) = T[h(\tau)] = -\frac{1}{2\pi}\int_{\leftrightarrows} h(t)\coth\frac{t-\tau}{2}\,dt$$

$$= -\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[h(t)\coth\frac{t-\tau}{2} - h(t+\pi i)\tanh\frac{t-\tau}{2}\right]dt, \qquad (14)$$

connected with Schwarz's integral for the strip $0 < \operatorname{Im} z < \pi$ (see problem **1086**), and

$$\frac{1}{2\pi i}\int_{\leftrightarrows} h(t)\coth\frac{t-z}{2}\,dt, \tag{15}$$

($\leftrightarrows$ denotes the boundary of the strip).

The function $\tilde{h} = T(h)$ is said to be conjugate to the function h.

1144. Prove the following assertions:

(1) The principal value of the integral (15) at the boundary point τ of the strip $0 < \operatorname{Im} z < \pi$, if it exists, equals $i\tilde{h}(\tau)$.

(2) If $h \equiv \text{const}$, then $\tilde{h} \equiv 0$.

(3) If $h(t) \in \operatorname{Lip}\alpha$ $(0 < \alpha \leqslant 1)$ on any finite interval and the integral (9) converges to $\pm\infty$ (for example, if $h(t)$ is bounded and $\int_{-\infty}^{\infty} |h(t)-h(t+\pi i)|\,dt < \infty$), then $\tilde{h}(\tau)$ exists:

$$\tilde{h}(\tau) = -\frac{1}{2\pi}\int_{\leftrightarrows} [h(t)-h(\tau)]\coth\frac{t-\tau}{2}\,dt$$

and is continuous (the stronger assertion: $\tilde{h}(\tau) \in \operatorname{Lip}\alpha$, if $0 < \alpha < 1$).

(4) If $h(t+\pi i) = \pm\tilde{h}(t)$, then $\tilde{h}(t+\pi i) = \mp\tilde{h}(t)$.

(5) If $h(t+\omega) = \pm h(t)$, then $\tilde{h}(t+\omega) = \pm\tilde{h}(t)$ (ω a real. number).

1145. Let $f(z)$ and $f_1(z)$ be analytic funetions defined by Schwarz's integral (15) in the strips $0 < \operatorname{Im} z < \pi$ and $-\pi < \operatorname{Im} z < 0$ respectively.

Prove that:

(1) $f_1(z) = -\overline{f(\bar{z})}$ $(-\pi < \operatorname{Im} z < 0)$.

(2) Schwarz's integral (15) defines in each of the strips $k\pi < \operatorname{Im} z < (k+1)\pi$ $(k = 0, \pm 1, \pm 2, \ldots)$ an analytic function such that if $F(z)$ is the value of the integral (15) at the point z and $k\pi < \operatorname{Im} z < (k+1)\pi$, then $F(z) = f(z-k\pi i)$, if k is even, and $F(z) = f[z-(k+1)\pi i]$, if k is odd.

(3) With the additional assumptions that the function $h(t)$ is bounded, belongs to $\operatorname{Lip}\alpha$ $(0 < \alpha \leqslant 1)$ uniformly on the boundary of the strip and

$$\int_{-\infty}^{\infty} |h(t)-h(t+\pi i)|\,dt < \infty:$$

we have

$$f(t) = \lim_{z \to t} f(z) = h(t) + i\tilde{h}(t),$$
$$f_1(t) = \lim_{z \to t} f_1(z) = -h(t) + i\tilde{h}(t) \qquad (\operatorname{Im} t = 0, \operatorname{Im} t = \pi).$$

1146. (1) Let $\phi(z)$ and $\psi(z)$ be functions analytic respectively in the strips $0 < \operatorname{Im} z < \pi$ and $-\pi < \operatorname{Im} z < 0$, bounded at $\pm\infty$ and continuous up to and including the edges of the strips. Prove, that if $\phi(t)$ and $\psi(\bar{t})$ ($\operatorname{Im} t = 0, \operatorname{Im} t = \pi$) belong to Lip α ($0 < \alpha \leqslant 1$) and there exist the transformations $T[\phi(t)]$, $T[\psi(\bar{t})]$ and the limits

$$\lim_{x \to \pm\infty} \frac{1}{\pi} \int_0^{\pi} \phi(x+iy)\,dy = \phi_{\pm\infty}, \quad \lim_{x \to \pm\infty} \frac{1}{\pi} \int_0^{\pi} \psi(x-iy)\,dy = \psi_{\pm\infty}.$$

then

$$T[\phi(t)] = -i\left[\phi(t) - \frac{\phi_\infty + \phi_{-\infty}}{2}\right], \quad T[\psi(\bar{t})] = i\left[\psi(\bar{t}) - \frac{\psi_\infty + \psi_{-\infty}}{2}\right].$$

(2) Find $T[\phi(t)]$ and $T[\psi(\bar{t})]$ ($\operatorname{Im} t = 0$, $\operatorname{Im} t = \pi$), if the functions $\phi(z)$ and $\psi(z)$ have poles at the point a ($0 < \operatorname{Im} a < \pi$) and at the point b ($-\pi < \operatorname{Im} b < 0$) respectively and elsewhere satisfy the conditions of part (1).

1147. Find functions defined by Schwarz's integral (15) in the strip $0 < \operatorname{Im} z < \pi$ for the following functions $h(t)$:

(1) $h(t) = \operatorname{Re} \phi(t)$, $\phi(z)$ is from problem **1146** (1);
(2) $h(t) = \operatorname{Re} \psi(\bar{t})$, $\psi(z)$ is from problem **1146** (1);
(3) $h(t) = \operatorname{Re} \phi(t)$, $\phi(z)$ is from problem **1146** (2);
(4) $h(t) = \operatorname{Re} \psi(\bar{t})$, $\psi(z)$ is from problem **1146** (2);
(5) $h(t) = \operatorname{Re} 1/(t-a)$, $0 < \operatorname{Im} a < \pi$;
(6) $h(t) = \operatorname{Re} \coth (t-a)$, $0 < \operatorname{Im} a < \pi$.

We consider below tho composition of singular integrals with the Cauchy kernel $1/(\zeta - z)$. A fundamental position is occupied by the Poincaré–Bertrand formula:

$$\int_C \frac{d\sigma}{\sigma - \tau} \int_C \frac{F(\zeta, \sigma)\,d\zeta}{\zeta - \sigma} = -\pi^2 F(\tau, \tau) + \int_C d\zeta \int_C \frac{F(\zeta, \sigma)\,d\sigma}{(\sigma - \tau)(\zeta - \sigma)},$$

where C is a smooth contour, closed or open (in the latter case a point $\tau \in C$ differs from an end point), the function $F(\zeta, \sigma) \in \mathrm{Lip}\,\alpha$ with respect to ζ and σ and if C goes off to infinity, decreases sufficiently rapidly at infinity (see the book: N. I. MUSKHELISHVILI, *Singular Integral Equations* (*Singulyarnyye integral'nyye uravneniya*), § 23, Gostekhizdat, 1946 or F. D. GAKHOV, *Boundary Problems* (*Kraevyye zadachi*) Chapter I, § 7, Fizmatgiz, 1958).

In problems **1148–1164** verify the Poincaré–Bertrand formula for the given functions $F(\zeta, \sigma)$ ($\zeta = x+iy$) by the actual evaluation of both sides of it. In these problems: C_1 is the closed contour enclosing the origin of coordinates, C_2 is the horizontal axis, C_3 is the segment $[0, 1]$ and C_4 is the semicircle $|\zeta| = R$, $\operatorname{Im}\zeta \geqslant 0$, $n = 0, 1, 2, \ldots$

1148. ζ^n, the contour C_1.

1149. $1/\zeta^n$, the contour C_1.

1150. σ^n, the contour C_1.

1151. $1/\sigma^n$, the contour C_1.

1152. $\zeta^p\sigma^q$, the contour C_1 ($p, q = 0, \pm 1, \pm 2, \ldots$).

1153. $1/(x+bi)$, the contour C_2.

1154. $\cos\lambda x \cos\mu\sigma$ ($\lambda > 0, \mu \geqslant 0$), the contour C_2.

1155. $\dfrac{\cos\lambda x}{\sigma+bi}$ ($\lambda > 0$), the contour C_2.

1156. x, the contour C_3.

HINT. Make use of the formula

$$\int_0^1 \log\frac{1-x}{x}\,\frac{dx}{x-t} = \frac{1}{2}\left(\log\frac{t}{1-t}\right)^2 - \frac{\pi^2}{2}.$$

See problem **1051** (1).

1157. $x\sigma$, the contour C_3.

1158. x^n, the contour C_3.

In problems **1159–1164** make use of the integrals:

$$\int_{C_4}\frac{d\sigma}{\sigma-\tau} = \log\frac{R+\tau}{R-\tau} = \log\left|\frac{R+\tau}{R-\tau}\right| + \frac{i\pi}{2},$$

$$\int_{C_4}\log\frac{R+\sigma}{R-\sigma}\,d\sigma = 0, \qquad \int_{C_4}\frac{1}{\sigma}\log\frac{R+\sigma}{R-\sigma}\,d\sigma = -\frac{\pi^2}{2},$$

$$\int_{C_4}\log\frac{R+\sigma}{R-\sigma}\,\frac{d\sigma}{\sigma-\tau} = \frac{1}{2}\left[\log^2\frac{R+\tau}{R-\tau} - \pi^2\right]$$

(for the last equality see the hint to problem **1051**).

1159. 1, the contour C_4.
1160. ζ, the contour C_4.
1161. σ, the contour C_4.
1162. ζ/σ, the contour C_4.
1163. σ/ζ, the contour C_4.
1164. $1/\sigma^2$, the contour C_4.

Singular integrals of the form

$$S\omega(z) = -\frac{1}{\pi}\iint\limits_E \frac{\omega(\zeta)\,d\xi\,d\eta}{(\zeta-z)^2} \tag{16}$$

are considered below, where $\zeta = \xi+i\eta$, extending to the entire ζ-plane and defined as the principal value

$$S\omega(z) = \lim_{\varrho\to 0} -\frac{1}{\pi}\iint\limits_{|\zeta-z|>\varrho} \frac{\omega(\zeta)\,d\xi\,d\eta}{(\zeta-z)^2}. \tag{17}$$

The transformation $S\omega$ is closely connected with the integral transformation

$$T\omega(z) = -\frac{1}{\pi}\iint\limits_E \frac{\omega(\zeta)\,d\xi\,d\eta}{\zeta-z}. \tag{18}$$

See problems **1058–1062.**

1165. Prove that if $\omega(\zeta)\in \mathrm{Lip}\,\alpha$ $(0<\alpha\leqslant 1)$ and the integral $\iint\limits_E |\omega(\zeta)|\,d\xi\,d\eta < \infty$, then the integral (16) exists and

$$S\omega(z) = -\frac{1}{\pi}\iint\limits_{|\zeta-z|>R} \frac{\omega(\zeta)\,d\xi\,d\eta}{(\zeta-z)^2} - \frac{1}{\pi}\iint\limits_{|\zeta-z|<R} \frac{\omega(\zeta)-\omega(z)}{(\zeta-z)^2}\,d\xi\,d\eta.$$

HINT. Prove that

$$\iint\limits_{\varrho<|\zeta-z|<R} \frac{d\xi\,d\eta}{(\zeta-z)^2} = 0.$$

1166. Let $\omega(z)\in C_0^2$ (twice continuously differentiable and equal to zero outside a compact set). Prove that

$$(T\omega)_z = S\omega,\quad (T\omega)_{\bar z} = \omega$$

(see problem **1061**), where the partial derivatives with respect to z and $\bar z$ denote as usual.

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right),\qquad \frac{\partial}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

1167. The functions $w(z)$ which effect the quasi-conformal mapping transforming the infinitesimal ellipses

$$E\,dx^2+2F\,dx\,dy+G\,dy^2 = ds^2$$

or, in a different notation,

$$|dz+q(z)d\bar{z}|^2 = ds^2, \quad |q(z)| < 1,$$

into infinitesimal circles, satisfy the equation

$$w_{\bar{z}} = q(z)w_z,$$

and conversely (see problem **378**). Show that if a solution of this equation is sought in the form

$$w(z) = T\omega(z)+\Phi(z),$$

where $\Phi(z)$ is an arbitrary integral function, and $\omega(z)$ is an unknown function, then for the definition of $\omega(z)$ there is obtained the singular integral equation

$$\omega - qS\omega = q\Phi'.$$

REMARK. For the development of this idea for the actual construction of $w(z)$ see the book: I. N. VEKUA, *Generalised Analytic Functions* (*Obobshchennyye analiticheskiye funktsii*), Fizmatgiz, 1959, English translation published by Pergamon Press (1962). References to the literature on this problem are given there.

1168. Prove that the transformation

$$S\omega, \ \omega \in L^2 \ \left(\text{that is, } ||\omega||^2 = \int\int_E |\omega(\zeta)|^2 d\xi d\eta < \infty\right),$$

is isometric, that is,

$$||S\omega|| = ||\omega||.$$

HINT. First prove the formula for $\omega \in C_0^2$ by integrating $||S\omega||^2$ by parts and using the relations of problem **1166**. Then prove the passage to L^2 from C_0^2 whieh is everywhere dense in L^2.

1169. Prove that if $\omega \in C_0^2$, then

$$SM_\varrho\omega = S_\varrho\omega. \tag{19}$$

where

$$M_\varrho\omega(\zeta) = \frac{1}{2\pi}\int_0^{2\pi} \omega(\zeta+\varrho e^{i\theta})d\theta,$$

$$S_\varrho\omega(z) = -\frac{1}{\pi}\int\int_{|\zeta-z|>\varrho} \frac{\omega(\zeta)d\xi d\eta}{(\zeta-z)^2},$$

and $S_\varrho \omega$ converges in mean to $S\omega$, that is $||S_\varrho \omega - S\omega|| \to \infty$ as $\varrho \to 0$.

HINT. Putting $t = \varrho e^{i\theta}$ and changing the order of integration in $S_\varrho \omega$ with the substitution $\zeta_1 = \zeta + \varrho e^{i\theta}$, use the fact that

$$\frac{1}{2\pi i} \int\limits_{|t|=\varrho} \frac{dt}{t(\zeta_1 - z - t)^2} = \begin{cases} \dfrac{1}{(\zeta_1 - z)^2}, & \text{if } |\zeta_1 - z| > \varrho, \\ 0, & \text{if } |\zeta_1 - z| < \varrho. \end{cases}$$

For the proof of the second asscrtion use the fact that (prove this)

$$||M_\varrho \omega - \omega|| \to 0 \text{ as } \varrho \to 0.$$

REMARK. The rclation (19) also holds almost everywhere for $\omega \in L^2$. This enables us to define for $\omega \in L^2$ the transformation $S\omega$ as the limit in mean of the transformation $S_\varrho \omega$.

1170. Let $H_\alpha = L^2 \cap \operatorname{Lip} \alpha$, that is, H_α is the space of functions which belong simultaneously to L^2 and Lip α. Prove that all functions $g(\zeta) \in H_\alpha$ satisfy the inequality

$$|Sg(1) - Sg(-1)| \leqslant A_\alpha |g|_\alpha,$$

where

$$A_\alpha = \frac{8}{\pi} \iint\limits_{\operatorname{Re}\zeta > 0} \frac{|\zeta - 1|^{\alpha-2} |\zeta|}{|\zeta + 1|^2} d\xi d\eta,$$

$$|g|_\alpha = \inf \frac{|g(\zeta_1) - g(\zeta_2)|}{|\zeta_1 - \zeta_2|^\alpha}.$$

HINT. Subtract from the difference which is being estimated multiplied by $-1/\pi$ the quantity

$$g(1) \iint\limits_{\operatorname{Re}\zeta > 0} \frac{4\zeta \, d\xi \, d\eta}{(\zeta^2 - 1)^2} + g(-1) \iint\limits_{\operatorname{Re}\zeta > 0} \frac{4\zeta \, d\xi \, d\eta}{(\zeta^2 - 1)^2}$$

(prove that both sides are equal to zero!).

1171. Let $\omega(\zeta) \in H_\alpha$ and

$$\tilde{\omega}(\zeta) = \omega\left(\frac{u - v}{2} \zeta + \frac{u + v}{2}\right).$$

Prove that

$$|\tilde{\omega}|_\alpha = 2^{-\alpha} |u - v|^\alpha |\omega|_\alpha$$

and using this, that

$$|S\omega(u) - S\omega(v)| \leqslant 2^\alpha A_\alpha |\omega|_\alpha |u - v|^\alpha.$$

REMARK. From this result it follows that S is a bounded linear operator in H_α.

1172. Let $g(\zeta) \in H_\alpha$ and

$$T_0 g(z) = -\frac{1}{\pi}\iint_E \left(\frac{1}{\zeta - z} - \frac{1}{\zeta}\right) g(\zeta)\,d\xi\,d\eta$$

(in comparison with Tg the passage to $T_0 g$ ensures integrability at infinity). Prove that

$$-\frac{1}{\pi}\iint_E \left(\frac{1}{\zeta - 1} - \frac{1}{\zeta}\right) d\xi\,d\eta = 1 \tag{20}$$

and using this fact, that

$$|T_0 g(1) - g(0) - Sg(0)| \leqslant B_\alpha |g|_\alpha,$$

where

$$B_\alpha = \frac{1}{\pi}\iint_E \frac{|\zeta|^{\alpha-2}}{|\zeta - 1|}\,d\xi\,d\eta.$$

HINT. Multiply $g(0)$ by the left side of (20).

1173. Let $\omega(\zeta) \in H_\alpha$. Putting

$$\tilde{\omega}(\zeta) = \omega[u + \zeta(u - v)],$$

prove that

$$|T_0\omega(u) - T_0\omega(v) - \omega(v)(\bar{u} - \bar{v}) - S\omega(v)(u - v)| \leqslant B_\alpha |\omega|_\alpha |u - v|^{1+\alpha},$$

and deduce from this the relation:

$$\left(T_0\omega(z)\right)_{\bar{z}} = \omega(z), \qquad \left(T_0\omega(z)\right)_z = S\omega(z).$$

1174. Show that the search for the quasi-conformal mapping $w(z)$ of the whole w-plane onto itself, which satisfies the equation

$$w_{\bar{z}} = q(z) w_z \qquad (q \in H_\alpha)$$

by means of the substitution

$$w(z) = z + T_0\omega(z)$$

leads to the singular integral equation

$$\omega - qS\omega = q,$$

(see the remark on problem **1167**).

CHAPTER IX

ANALYTIC CONTINUATION. SINGULARITIES OF MANY-VALUED CHARACTER. RIEMANN SURFACES

§ 1. Analytic continuation

1175. The function $f(z) = \sum_{n=0}^{\infty} z^n$ is expanded into a Taylor series in the neighbourhood of the point $z = a$ ($|a| < 1$). For what values of a does this expansion permit the function $f(z)$ to be continued analytically?

1176. The sum of the power series $f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n}$ is expanded into a Taylor series in the neighbourhood of the point $z = -1/2$. What is the domain into which the function $f(z)$ is continued by this?

1177. Prove that the function $f(z) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^n}{n}$ can be continued into a larger domain by means of the series

$$\log 2 - \frac{1-z}{2} - \frac{(1-z)^2}{2 \cdot 2^2} - \frac{(1-z)^3}{3 \cdot 2^3} - \dots .$$

1178. The power series

$$f_1(z) = \sum_{n=1}^{\infty} \frac{z^n}{n} \quad \text{and} \quad f_2(z) = i\pi + \sum_{n=1}^{\infty} (-1)^n \frac{(z-2)^n}{n}$$

have no common domain of convergence. Prove that the functions $f_1(z)$ and $f_2(z)$ are nevertheless analytic continuations of one another.

1179. Prove that the functions defined by the series

$$1 + az + a^2z^2 + \dots \quad \text{and} \quad \frac{1}{1-z} - \frac{(1-a)z}{(1-z)^2} + \frac{(1-a)^2z^2}{(1-z)^3} - \dots ,$$

are analytic continuations of one another.

1180. Let the power series

$$f(z) = a_0+a_1z+ \dots +a_nz^n+\dots$$

have the radius of convergence $R = 1$. By carrying out the change of variable $z = Z/(1+Z)$ let us transform it into the form

$$f(z) = f[Z/(1+Z)] = F(Z) = c_0+c_1Z+ \dots +c_nZ^n+ \dots$$

Denoting by ϱ, the radius of convergence of the series obtained prove the following assertions:

(1) $\varrho \geqslant \frac{1}{2}$, and if the point $z = -1$ is a singularity of the function $f(z)$, then $\varrho = \frac{1}{2}$.

(2) If $\frac{1}{2} < \varrho < 1$, then the equality $f(z) = F(Z) = F\left(\frac{z}{1-z}\right)$ permits the analytic continuation of the function $f(z)$ into a domain exterior to the disk $|z| < 1$ and interior to the circle of Apollonius $\left|\frac{z}{z-1}\right| = \varrho$.

(3) If $\varrho = 1$, the equality given in part 2 analytically continues the function $f(z)$ into the half-plane $\operatorname{Re} z < 1/2$.

(4) If $\varrho > 1$, then the function $f(z)$ is analytically continuable into a domain exterior to the circle of Apollonius $\left|\frac{z}{z-1}\right| = \varrho$.

1181. Prove that the power series $f(z) = \sum_{n=0}^{\infty} z^{2^n}$ represents a function, analytic in the disk $|z| < 1$ and having the circle $|z| = 1$ as its natural boundary (that is, $f(z)$ is a function which is not continuable beyond the unit circle).

HINT. Using the identity

$$f(z) = z^2+z^4+ \dots +z^{2^k}+f(z^{2^k}),$$

prove that for any point of the form $\zeta = \sqrt[2^k]{1}$ (k is a natural number) $f(t\zeta) \to \infty$ as $t \to 1$ $(0 < t < 1)$.

In problems **1182–1184** prove that the functions represented by the given power series are not continuable beyond the unit circle.

1182. $f(z) = \sum_{n=0}^{\infty} \frac{z^{2^n}}{2^n}$.

HINT. Uso Pringsheim's theorem (see problem **642**).

1183. $f(z) = \sum_{n=0}^{\infty} z^{n!}$.

HINT. If p and q are coprime integers and $n \geqslant q$, then $\left(re^{\frac{2p\pi i}{q}}\right)^{n!} = r^{n!}$.

1184. $f(z) = \sum_{n=0}^{\infty} \frac{z^{n!}}{n^2}$.

REMARK. In problems **1182–1184** particular cases are considered of Hadamard's general gap theorem:

If the indices of non-zero coefficients of the power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ form a sequence $n_1, n_2, \ldots$, in which $n_{k+1} > (1+\alpha)n_k$ where $\alpha > 0$, then the boundary of the circle of convergence of the series is a natural boundary of the function $f(z)$.

1185. Prove that the series $\sum_{n=1}^{\infty}\left(\frac{1}{1-z^{n+1}} - \frac{1}{1-z^n}\right)$ in the domains $|z| < 1$ and $|z| > 1$ represents two analytic functions which are not analytic continuations of one another (see also problems **696–699**).

1186. Let $f(z)$ and $\phi(z)$ be arbitrary integral functions, and $S(z) = \sum_{n=1}^{\infty}\left(\frac{1-z^n}{1+z^n} - \frac{1-z^{n-1}}{1+z^{n-1}}\right)$. Prove that the expression

$$\psi(z) = \frac{1}{2}[f(z)+\phi(z)]+\frac{1}{2}S(z)[f(z)-\phi(z)]$$

represents in the domain $|z| < 1$ the function $f(z)$ and in the domain $|z| > 1$ the function $\phi(z)$.

1187. (1) Prove that if α is a real irrational number the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n(z-e^{2in\pi\alpha})}$$

represents in the domains $|z| < 1$ and $|z| > 1$ analytic functions for each of which the circle $|z| = 1$ is a natural boundary.

HINT. Prove that the sum of the series increases without limit as $z \to e^{2im\alpha\pi}$ along a radius vector.

REMARK. This problem is a particular case of the following general theorem:

Let L be a curve, closed or open, which has at every point a definite radius of curvature. If the series $\sum_{n=1}^{\infty} c_n$ converges absolutely and the points $a_1, a_2, \ldots, a_n, \ldots$ all lie on the curve L and are distributed on it so that any finite arc of the curve L always contains an infinite set of them, then the series

$$F(z) = \sum_{n=1}^{\infty} \frac{c_n}{a_n - z}$$

represents a function analytic in any domain not containing points of the curve L, and for which this curve is a singular line (see GOURSAT, *A Course in Mathematical Analysis*, Vol. 2, Chapter XVI).

(2) Prove that if α is a real rational number, then the series of part (1) represents a rational function.

1188. Prove that the series $\sum_{n=1}^{\infty} \frac{1}{n^z + n}$ $(n^z = e^{z \log n})$ converges for $\operatorname{Re} z > 1$ and its sum has the straight line $\operatorname{Re} z = 1$ as its natural boundary.

1189. Prove that the function $f(z) = \sum_{m=0}^{\infty} e^{-m! z}$ is analytic for $\operatorname{Re} z > 0$ and has the straight line $\operatorname{Re} z = 0$ as its natural boundary.

1190. Prove that the function $f(z)$ defined in the half-plane $\operatorname{Re} z > 0$ by the Dirichlet series

$$f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$$

where $a_n = (-1)^{n+1}$, $\lambda_{2k-1} = 2k$, $\lambda_{2k} = 2k + e^{-2k}$ $(k = 1, 2, \ldots)$, can be continued analytically into the half-plane $\operatorname{Re} z > -1$.

HINT. Write $f(z)$ in the form

$$f(z) = \sum_{k=1}^{\infty} (1 - e^{-ze^{-2k}}) e^{-2kz}$$

and prove that in any finite domain belonging to the half-plane $\operatorname{Re} z > -1$,

$$|1 - e^{-ze^{-2k}}| < Me^{-2k},$$

where M is a constant for the domain considered.

REMARK. The given problem shows that on the straight line bounding the half-plane of convergence the sum of a Dirichlet series may not have singular points.

1191*. The function $f(s)$ defined in the half-plane $\operatorname{Re} s > 0$ by means of the Laplace integral

$$f(s) = \int_0^\infty e^{-st} e^t \sin e^t \, dt,$$

is to be continued analytically into the half-plane $\operatorname{Re} s > -1$.

1192. Euler's Gamma function is defined in the half-plane $\operatorname{Re} z > 0$ by means of the integral

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \qquad (t^{z-1} = e^{(z-1)\log t}).$$

Integrating by parts the right-hand side of this equation, show that the function $\Gamma(z)$ is continued analytically to the whole plane as a meromorphic function with the simple poles $0, -1, -2, \ldots, -n, \ldots$, the residue at the pole $-n$ being equal to $\dfrac{(-1)^n}{n!}$.

1193. Show that it is possible to continue the Gamma function analytically by means of the formula

$$\Gamma(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(z+n)} + \int_1^\infty e^{-t} t^{z-1} dt.$$

HINT. In the integral $\int_0^1 e^{-t} t^{z-1} dt$ replace the function e^{-t} by its expansion into a power series.

1194. In problem **420** it was proved that for $0 < x < 1$

$$\int_0^\infty t^{x-1} \cos t \, dt = \Gamma(x) \cos \frac{\pi x}{2}, \qquad \int_0^\infty t^{x-1} \sin t \, dt = \Gamma(x) \sin \frac{\pi x}{2}.$$

In what domains of the z-plane are the given formulae valid?

1195. Prove that $\Gamma(z)$ can be continued to the whole of its domain of existence by means of the formula

$$\Gamma(z) = \frac{i}{2 \sin \pi z} \int_C e^{-w} (-w)^{z-1} dw \qquad ((-w)^{z-1} = e^{(z-1)\log(-w)}),$$

where the contour C consists of a cut along the positive part of the real axis, the detour round the coordinate origin being traversed anticlockwise.

1196. Let $\zeta(z)$ be Riemann's Zeta function (see problem **695**)

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \qquad (\operatorname{Re} z > 1).$$

Prove that for $\operatorname{Re} z > 1$

$$\zeta(z) = \frac{1}{\Gamma(z)} \int_0^{\infty} \frac{w^{z-1} dw}{(e^w - 1)}$$

and obtain from this the analytic continuation of the function $\zeta(z)$ to the whole plane, excluding the point $z = 1$; explain the character of the singularity of the function $\zeta(z)$ at the point $z = 1$.

HINT. For the analytic continuation consider the integral $\int_C \frac{(-w)^{z-1}}{e^w - 1} dw$ where C is the contour of problem **1195**.

1197*. Let the function $f(z)$ be expanded into the power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ which has the radius of convergence $R = 1$. Let us denote by $\phi(z)$ the sum of the series $\phi(z) = \sum_{n=0}^{\infty} \frac{a_n z^n}{n!}$ (the function $\phi(z)$ is the Borel transform of the function $f(z)$ — it is an integral function, see problem **535**).

Prove that for $|z| < 1$ the equality

$$\int_0^{\infty} e^{-t} \phi(zt) dt = f(z)$$

is satisfied. Prove also, that the function $\int_0^{\infty} e^{-t} \phi(zt) dt$ effects the analytie continuation of the function $f(z)$ into the domain G defined as follows: A straight line is drawn through every singularity of the function $f(z)$, perpendicular to the segment eonnecting this singularity to the coordinate origin: G is the eonvex domain containing the eirele $|z| < 1$, the boundary of which consists of points of these straight lines; if the number of these straight lines is finite, then G is a polygon (Borel's method of continuation).

1198. Verify Borel's method of continuation for the following series:

$$(1)\ \sum_{n=0}^{\infty} z^n; \qquad (2)\ \sum_{n=0}^{\infty} z^{2n}; \qquad (3)\ \sum_{n=0}^{\infty} z^{4n}.$$

1199. In the integral of Cauchy type

$$F(z) = \frac{1}{2\pi i}\int_C \frac{\phi(\zeta)\,d\zeta}{\zeta - z}$$

let C be a simple closed contour and $\phi(\zeta)$ a function continuous along C. Prove that in order that one of the functions $F^+(z)$ and $F^-(z)$ (see page 153) should be the analytic continuation of the other through an arc $\gamma \in C$, it is necessary and sufficient that $f(z) \equiv 0$ on the arc γ.

1200. Prove that if the function $\phi(\zeta)$ is not analytic† at even one point of the simple open arc C, then all the points of the arc C are singularities of the integral of Cauchy type

$$\frac{1}{2\pi i}\int_C \frac{\phi(\zeta)\,d\zeta}{\zeta - z}.$$

HINT. Commence with Sokhotskii's formulae for the limiting values of a Cauchy type integral.

1201. Let γ be a simple closed contour traversed in the positive direction, consisting of arcs γ_1, γ_2 with common end points z_1 and z_2 (Fig. 31), let G^+ be the domain within γ, G^- the domain outside γ.

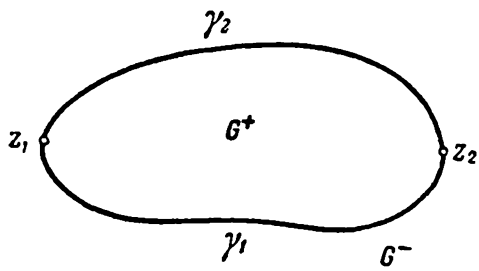

FIG. 31

† That is, there does not exist any analytic function identical with $\phi(\zeta)$ on any arc belonging to C.

In addition, let $F(z) = \int_{\gamma} \frac{\phi(\zeta)\,d\zeta}{\zeta - z}$ where $\phi(\zeta) = a$ if $\zeta \in \gamma_1$, and $\phi(\zeta) = b$, if $\zeta \in \gamma_2$ (a and b are complex constants). Find the functions $F^+(z)$ and $F^-(z)$ and continue analytically the function $F^-(z)$ into the domain G^+:

(a) through the arc γ_1; (b) through the arc γ_2.

1202. Let G be a doubly connected domain bounded by an interior contour γ and an exterior contour Γ, and $\phi(z)$ a function analytic in the closed region $G+\gamma+\Gamma$.

Prove that the function

$$\phi(z) - \frac{1}{2\pi i} \int_{\Gamma} \frac{\phi(\zeta)}{\zeta - z}\,d\zeta$$

is analytically continuable to the whole of the exterior of the contour γ, and the function

$$\phi(z) + \frac{1}{2\pi i} \int_{\gamma} \frac{\phi(\zeta)}{\zeta - z}\,d\zeta$$

to the whole of the interior of the contour Γ. The contours γ and Γ are traversed anticlockwise.

§ 2. Singularities of many-valued character. Riemann surfaces†

An isolated branch point $z = a$ of order $k-1$ of the function $w(z)$ (k is a natural number, $k \geqslant 2$) is characterised by the fact that there exists a branch of $w(z)$, which in the neighbourhood of the point $z = a$ admits of the representation

$$w = \sum_{-\infty}^{\infty} c_n (z-a)^{\frac{n}{k}} \qquad (a \neq \infty)$$

or

$$w = \sum_{-\infty}^{\infty} c_n z^{-\frac{n}{k}} \qquad (a = \infty).$$

If only a finite number of the coefficients c_n with negative subscripts differ from zero the point $z = a$ (or $z = \infty$) is called an *algebraic branch point*. In the contrary case the corresponding point is called a *transcendental branch point* (an essential singularity of many-valued character).

Above one and the same point of the z-plane the function $w(z)$ may have not more than an enumerable set of different algebraic and transcendental

† For this section see [1, Chapter VIII]; V. V. Golubev *Lectures on the Analytic Theory of Differential Equations*, (*Lektsii po analiticheskoi teorii differential'nykh uravnenii*), Gostekhizdat, 1951; R. Nevanlinna, *Uniformierung*, Berlin, 1953.

branch points, regular points and singularities of single-valued character. On the Riemann surface of the function $w(z)$ above the z-plane such points have neighbourhoods which do not intersect.

In every such neighbourhood w is a single-valued function of the parameter t:

$$w = \sum_{n=-N}^{\infty} c_n t^n,$$

where

$$t = \begin{cases} (z-a)^{\frac{1}{k}} & (a \neq \infty), \\ z^{-\frac{1}{k}} & (a = \infty). \end{cases}$$

A point $z = a$ or $z = \infty$ is said to be a *logarithmic branch point* if it is such that any branch $w(z)$ admits of unbounded analytic continuation in the domain $0 < |z-a| < r$ (or $R < |z| < \infty$) and is infinitely many valued there. Such a branch $w(z)$ in the neighbourhood of a logarithmic branch point becomes a single-valued analytic function on passing to the parameter $t = \operatorname{Log}(z-a)$, $\operatorname{Re} t < \varrho$ (or $t = \operatorname{Log} z$, $\operatorname{Re} t > \varrho$). It must be remembered that on the Riemann surface above one and the same point z, in addition to various logarithmic branch points there may also be found other points of single-valued and many-valued character.

1203. Explain for which values of z the values of $w(z)$ are identical on all the sheets of its Riemann surface above the z-plane, if:

(1) $w = (z^2 - 9)\sqrt[3]{(z)}$;

(2) $w = \sin z + (z^2 + 4)\operatorname{Log} z$;

(3) $w = \sin z + (z^2 + 4)^2 \operatorname{Log} z$.

Are the values of $w'(z)$ identical at the same points?

1204. For each of the functions:

(1) $w = z\sqrt{z}$.

(2) $w = z^2 \operatorname{Log} z$, $w(0) = 0$,

verify that at the point $z = 0$ there exists a first derivative which is the same for all the branches and that a finite second derivative does not exist.

In problems **1205–1212** expand each of the given functions $w(z)$ into a series of powers of the local parameter t in the neighbourhood of all the points of its Riemann surface situated above the given z points; indicate the domains of convergence of the resulting series.

1205. $w = \dfrac{1}{1+\sqrt{(2-z)}}$, $z = 1$, $z = 2$.

1206. $w = \sqrt{[\sqrt{(z-1)}-2]}$, $z = 1$, $z = 5$, $z = \infty$.

1207. $w = \sqrt{[1+\sqrt{(z-1)}]}$, $z = 1$.

1208. $w = \sqrt{[(\sqrt{z}-a)(\sqrt{z}-b)]}$, $z = \infty$ $(a \neq b)$.

1209. $w = e^{\frac{1}{\sqrt{z}}}$, $z = 0$.

1210. $w = \dfrac{\sin\sqrt{z}}{z^2}$, $z = 0$.

1211. $w = \cot\sqrt[3]{z}$, $z = 0$.

1212. $w = \sqrt{\sin z}$, $z = 0$.

If the function $w = f(z)$ is single valued and the function $z(w)$ inverse to it is many valued, for the determination of the algebraic branch points of the function $z(w)$ it is necessary to find the zeros of $f'(z)$, the multiple poles of $f(z)$ and investigate the behaviour of $f(z)$ at infinity. In this, a point $z_0 \neq \infty$ corresponds to an algebraic branch point of order $k-1$ of the function $z(w)$, if in the neighbourhood of z_0 the Laurent expansion of the function $f(z)$ has the form

$$f(z) = w_0 + \sum_{n=k}^{\infty} c_n (z-z_0)^n \qquad (c_k \neq 0)$$

or

$$f(z) = \sum_{n=-k}^{\infty} c_n (z-z_0)^n \qquad (c_{-k} \neq 0).$$

If $z_0 = \infty$, then the given expansions must be of the form

$$f(z) = \sum_{n=k}^{-\infty} c_n z^n \qquad (c_k \neq 0)$$

or

$$f(z) = w_0 + \sum_{n=-k}^{-\infty} c_n z^n \qquad (c_{-k} \neq 0).$$

In problems **1213–1219** determine the singularities of $z(w)$ if w is the given function.

1213. $w = z(1-z)$.

1214. $w = z^3 - 3z$.

1215. $w = \dfrac{z}{(1+z)^2}$.

1216. $w = \left(\dfrac{z}{1-z}\right)^2$.

1217. $w = \dfrac{z-a}{z^2-1}$ $(0 < a < 1)$.

1218. $w = P_n(z)$ (a polynomial of degree n).

1219. $w = R(z)$ (a rational function).

In problems **1220–1228** investigate the mapping effected by the function $w(z)$, construct the Riemann surface R over the w-plane and subdivide the z-plane into domains corresponding to the sheets or half-sheets of R.

1220. $w=\left(1+\frac{z}{n}\right)^n$. Consider the limiting case $n \to \infty$.

1221. $w=\left(\frac{z-a}{z-b}\right)^n$.

1222. $w=\frac{1}{2}\left(z+\frac{1}{z}\right)$.

1223. $w=\frac{z}{(1-z)^2}$.

1224. $w=\frac{1}{2}\left(z^n+\frac{1}{z^n}\right)$. Find also the group of linear transformations, with respect to which the function w is invariant and explain which transformations of the Riemann surface correspond to the transformations of the group.

1225. $w=\frac{z}{(1+z^n)^2}$.

1226. $w=z-\frac{z^n}{n}$.

1227. $w=\frac{1}{z}+\frac{z^n}{n}$.

1228. $w=T_n(z)=\frac{1}{2^{n-1}}\cos(n\cos^{-1}z)$, $n \geqslant 1$ ($T_n(z)$ are Chebyshev polynomials).

In problems **1229–1232** find the singularities of the functions inverse to those given.

1229. $w=e^{\frac{1}{z}}$.

1230. $w=e^{t\left(z-\frac{1}{z}\right)}$ (t is a complex number).

1231. $w=\cos z+\sin z$.

1232. $w=\frac{\sin z}{z}$.

In problems **1233–1241** construct the Riemann surfaces above the w-plane.

1233. $w=\cos z$.

1234. $w=\sin z$.

1235. $w=\tan z$.

1236. $w=\cot z$.

1237. $w=\cosh z$.

1238. $w=\sinh z$.

1239. $w = \tanh z$.

1240. $w = \coth z$.

1241. $w = z + e^z$.

1242. Assuming that the Riemann surface above the w-plane of the rational function $w = R(\zeta)$ is known, construct the Riemann surfaces of the functions $z(w)$, if:

(1) $w = R(e^z)$; (2) $w = R(\sin z)$.

In problems **1243–1248** construct the Riemann surfaces of the given functions (above the z-plane).

1243. (1) $w = \sqrt{[(z-a)(z-b)]}$; (2) $w = \sqrt{[(z-a)(z-b)(z-c)]}$;

(3) $w = \sqrt[n]{\left(\prod_{k=1}^{n}(z-a_k)\right)}$ (consider separately the cases of even and odd n).

1244. (1) $w = \sqrt[3]{(z-a)}$; (2) $w = \sqrt[3]{[(z-a)(z-b)]}$;

(3) $w = \sqrt[3]{[(z-a)(z-b)(z-c)]}$; 4) $w = \sqrt[3]{\left[\prod_{k=1}^{n}(z-a_k)\right]}$, $n > 3$.

1245. $w = \sqrt[n]{[(z-a)(z-b)(z-c)]}$, $n > 3$.

1246. $w = \sqrt[3]{\left(\frac{z-a}{(z-b)^2}\right)} + \sqrt{(z-c)}$.

1247. $w = \sqrt{[\sqrt[3]{(z)}-1]}$.

1248. $w = \sqrt{(\sin z)}$.

In problems **1249–1254** investigate the mappings and construct the Riemann surfaces for the algebraic functions $w(z)$ and $z(w)$.

1249. $w^2+z^2 = 1$.

1250. $w^2 = \frac{z}{(1-z)^2}$.

HINT. In problem **1250**, use the solution of problem **1223**.

1251. $w^3+z^3-3wz = 0$.

HINT. Use the parametric representation

$$z = \frac{3t}{1+t^3}, \quad w = \frac{3t^2}{1+t^3}.$$

1252. $w^2 = \frac{z^3}{1-z}$.

HINT. Use the parametric representation $z = t^2/(1+t^2)$, $w = t^3/(1+t^2)$.

1253. $w^2 = z^2 \dfrac{1+z}{1-z}$.

HINT. Put $z = \dfrac{t^2-1}{t^2+1}$, $w = t\dfrac{t^2-1}{t^2+1}$.

1254. $w^n = \dfrac{z^n}{(1+z^n)^2}$.

HINT. Consider separately the cases of even and odd n.

If $z = a$ is a singular point of one of the branches $f(z)$ of a function $w(z)$ then the *set of indeterminateness* of $f(z)$ at the point $z = a$ is the set of the limiting values of $f(z)$ obtained from its values in the neighbourhood of $z = a$ above $|z - a| < r$ as $r \to 0$. For algebraic branch points and poles the set of indeterminateness consists of a single point. If the function is single valued and the point a is an isolated essential singularity, then by the Casorati–Weierstrass theorem the set of indeterminateness covers the whole plane (see also problem **638**). For transcendental and logarithmic branch points, and also for non-isolated essential singularities the set of indeterminateness may have a more complicated structure (see the book: V. V. GOLUBEV, *Lectures on the Analytic Theory of Differential Equations* (*Lektsii po analiticheskoi teorii differentsial'nykh uravnenii*), Chapter I, § 7, Gostekhizdat, 1951.

In problems **1255–1266** determine the singularities of the functions and also find the sets of indeterminateness at the transcendental and logarithmic branch points.

1255. $w = (\sqrt[6]{z})^4$.

1256. $w = (\sqrt[m]{z})^n$ (n, m are natural numbers).

1257. $w = e^{1/(\sqrt{z}-1)}$

1258. $w = (\sin\sqrt{z})/\sqrt{z}$.

1259. $w = \tan\dfrac{1-\sqrt{z}}{1+\sqrt{z}}$.

1260. $w = e^{(\operatorname{Log} z)^n}$ (n is an integer).

1261. $w = z^i = e^{i \operatorname{Log} z}$.

1262. $w = \sin \operatorname{Log} z$.

1263. $w = \dfrac{1}{z} \operatorname{Log} \dfrac{1}{1-z}$.

1264. $w = z + \operatorname{Log} z$.

1265. $w = \dfrac{1}{z} \sin^{-1} z$.

1266. $w = \dfrac{1}{z^2} + \cot^{-1} z$.

In problems **1267–1270** construct the Riemann surfaces of the given functions.

1267. $w = z^a$ (a is a complex number).

1268. $w = \operatorname{Log}[(z-a)(z-b)]$.

1269. $w = \operatorname{Log}[(z-a)(z-b)(z-c)]$.

1270. $w = \operatorname{Log} \sin z$.

1271. Let $\zeta = \phi(z)$ be a single-valued or many-valued analytic function, $R_{z,\phi}$ its Riemann surface above the z-plane and $w = f(\zeta)$ a single-valued analytic function with set of indeterminateness G_ζ. In what domains on $R_{z,\phi}$ does each of the given expressions:

(1) $w(z) = f[\phi(z)]$, (2) $w(z) = f(z) + \phi(z)$, (3) $w(z) = f(z)\phi(z)$

determine a unique analytic function?

In particular, consider the case when ϕ is algebraic or inverse to meromorphic, and f is a rational or transcendental meromorphic function.

In problems **1272–1302** explain which of the given functions $w(z)$ can be resolved into distinct algebraic functions and which not; determine also their singularities and where indicated construct their Riemann surfaces (n and m are natural numbers).

1272. $w = \sqrt{(z^2)}$ (compare with $(\sqrt{z})^2$).

1273. $w = \sqrt[6]{(z^4)}$ (compare with $(\sqrt[6]{z})^4$).

1274. $w = \sqrt[n]{(z)^m}$ (compare with $(\sqrt[n]{z})^m$).

1275. $w = \sqrt[n]{e^z}$.

1276. $w = \sqrt[n]{(\sin z)}$.

1277. $w = \operatorname{Log} z^n$.

1278. $w = \operatorname{Log} e^z$.

1279. $w = \operatorname{Log}\left(z - \dfrac{1}{z}\right)$.

1280. $w = \operatorname{Log}(e^z - 1)$. Construct the Riemann surface.

1281. $w = \operatorname{Log} \sin z$. Construct the Riemann surface.

1282. $w = \operatorname{Log} \tan z$. Construct the Riemann surface.

1283. $w = \cos^{-1}(\cos z)$ (compare with $\cos(\cos^{-1} z)$).

1284. $w = \tan^{-1}(\tan z)$ (compare with $\tan(\tan^{-1} z)$).

1285. (1) $w = (z^{r_1})^{r_2}$ (r_1, r_2 are rational numbers). Compare with $(z^{r_2})^{r_1}$. Consider, in particular, $r_1 = 2/3$, $r_2 = 3/2$;

(2) $w = z^{r_1} z^{r_2}$; (3) $w = z^{r_1} + z^{r_2}$.

1286. $w = \sqrt[n]{z}\,\sqrt[m]{(1-z)}$.

1287. $w = \sqrt[n]{[\sqrt[m]{(z)} - 1]}$.

1288. $w = \dfrac{\sqrt{z}}{\sqrt[4]{(z)} + 1}$.

1289. $w = \sqrt[n]{(\operatorname{Log} z)}$. Construct the Riemann surface.

1290. $w = \operatorname{Log} \operatorname{Log} z$. Construct the Riemann surface.

1291. $w = [\operatorname{Log}(z - 1)]^i$.

1292. $w = \sqrt[n]{(\sin^{-1} z)}$. Construct the Riemann surface.

1293. $w = \sin^{-1} \operatorname{Log} z$. Construct the Riemann surface.

1294. $w = \operatorname{Log} (\sqrt[n]{(z)} - 1)$. Construct the Riemann surface.

1295. $w = \operatorname{Log} [(1-\sqrt{z})/(1+\sqrt{z})$.

1296. $w = \sin^{-1}[\sqrt{z}/(1+z)]$.

1297. $w = \operatorname{Log} z^{\alpha}$ (α is a real number).

1298. $w = \sqrt{(z)} + \operatorname{Log} z$. Construct the Riemann surface.

1299. $w = \operatorname{Log} z + \operatorname{Log} z$.

1300. $w = \sin^{-1} z + \cos^{-1} z$.

1301. $w = \tan^{-1} z + \cot^{-1} z$.

1302. $w = \tanh^{-1} z - \coth^{-1} z$.

1303. Construct the Riemann surface of the function $w = (\operatorname{Log} z)^i$ and investigate the set of limiting values of w for one logarithmic branch point above the point $z = 0$, obtained for: (1) $r \to 0$, $\phi = \text{const}$; (2) $r \to 0$, $\alpha < \phi < \beta$; (3) $r = \text{const}$, $\phi \to \pm \infty$; (4) $r \to 0$, $\phi \to \pm \infty$.

1304. Let $\chi(z)$ be a single-valued analytic function in the circle $|z| < 1$, which cannot be continued beyond the circle $|z| = 1$.

Explain for what values of a the given functions resolve into distinct analytic functions, and for what values they do not.

(1) $w = \chi(z) + \sqrt[n]{(z-a)}$; (2) $w = \chi(z) \operatorname{Log} (z-a)$;
(3) $w = \chi(a+z^n)$; (4) $w = \chi(a+e^z)$.

1305. Explain the problem of the factorisation of the functions:

(1) $w = \sqrt[n]{(\zeta-a)}$ (2) $w = \operatorname{Log}(\zeta-a)$, where $\zeta = \chi^{-1}(z)$ and χ is the function of problem **1304.**

1306. Investigate the behaviour of the separate analytic functions determined by the equations:

(1) $w = \chi(z)(\operatorname{Log} z)^i$; (2) $w = \chi(z)[\operatorname{Log} (z-1)]^i$, where $\chi(z)$ is the function of problem **1304.**

Find, in particular, the set of indeterminateness in the neighbourhood of the logarithmic branch point.

HINT. Use the solution of problem **1303.**

1307. Let $f(z)$ be an integral function.

Construct the Riemann surfaces of the functions:

(1) $w = \sqrt{(f(z))}$; (2) $w = \operatorname{Log} f(z)$; (3) $w = [f(z)]^{\alpha}$ (α is an irrational number).

1308. Construct the Riemann surface of the function

$$w = \sqrt{\left[\sum_{n=1}^{\infty} z^{n!}\right]}.$$

1309. Let

$$f(z) = z + \sum_{n=1}^{\infty} \frac{z^{2^n}+2}{(2^n+1)(2^n+2)}.$$

Construct the Riemann surfaces of the functions:

(1) $w = \sqrt{(f(z))}$;

(2) $w = \operatorname{Log} f(z)$;

(3) $w = \operatorname{Log} f\left(\frac{z}{2}\right) + \operatorname{Log} f\left(\frac{1}{2z}\right)$.

HINT. First prove that the function $f(z)$ is sehlicht in the disk $|z| < 1$ and has the circle $|z| = 1$ as its natural boundary.

§ 3. Some classes of analytic functions with non-isolated singularities†

Let E be a closed set of points with connected complement Ω in the extended z-plane. Let AB, AD and AC be classes of single-valued analytic functions in Ω, respectively: bounded (the class AB), having a bounded Dirichlct integral with respect to Ω (the class AD), eontinuous in the extended z-plane (the class AC). If the set E is in some respect small it can be expected that from the analyticity of the function in Ω and the additional limitations connected with the class of function, it would follow that the funetion in E is analytic. Then the function is a constant in virtue of Liouville's theorem. In this ease the set E is said to be a *nul set* of the class of functions considered.

The classes of nul sets corresponding to the given classes of functions are denoted by N_{AB}, N_{AD}, N_{AC}, and the class of their complements in the domain Ω by O_{AB}, O_{AD}, O_{AC}.

A set E is said to be AB-removable in the domain G, eontaining E, if from the analyticity and boundedness in $G-E$ there follows analyticity everywhere in G (if G is identical with the extended z-plane, then AB-removableness in G is equivalent to the fact that $E \in N_{AB}$). AD-removableness and AC-removableness of a set E in the domain G are similarly defined (in the case of AC-removableness, functions are eonsidered of the class AC in G, that is, analytic in $G-E$ and continuous everywhere in G). For brevity F denotes any of the classes AB, AD, AC, if it is a question of some property common to them.

1310. Prove that the set E with connected complement Ω, which is the set of singularities of a single-valued function analytic in Ω, is closed.

1311. Prove that the isolated singular points of functions of the classes AC, AB, AD are removable and the sets of singularities of the functions of these classes are perfect.

† For this section see L. AHLFORS, A. BEURLING, (1950), Conformal invariants and function theoretic null-sets, *Acta Math.* vol. 83, 100–129.

HINT. For $f \in AD$ consider as $\varrho \to 0$ the Dirichlet integral $\iint_{\varrho<|z|<\varrho_0} |f'|^2 r\,dr\,d\phi$ (a singularity at zero), starting from the Laurent expansion

$$f(z) = \sum_{-\infty}^{\infty} c_n z^n.$$

1312. Let G be an arbitrary domain in the extended z-plane and $E_k (k = 1, 2, \ldots, q)$ bounded closed sets with connected complements Ω_k, located in G, no two of which have points in common:

$$E_k \subset G, \quad E_i \cap E_j = 0 \qquad (i \neq j).$$

Prove that the function $f(z)$, single-valued and analytic in $G - E$ $E = \bigcup_{k=1}^{q} E_k$, can be represented uniquely in the form

$$f(z) = \phi(z) + \sum_{k=1}^{q} \psi_k(z),$$

where $\phi(z)$ is analytic in G, $\psi_k(z)$ is analytic in Ω_k and $\psi_k(\infty) = 0$. In particular, if G is the extended z-plane, then

$$f(z) = f(\infty) + \sum_{k=1}^{q} \psi_k(z)$$

HINT. Separate the sets E_k in G by contours γ_k $(k = 1, 2, \ldots, q)$ and in the domain cut off from the variable domain G_n, converging to G, by the contours γ_k, represent $f(z)$ by Cauchy's formula.

REMARK. $\psi_k(z)$ has on E_k the same singularities as $f(z)$ and is the analogue of the principal part of the Laurent expansion in the neighbourhood of an isolated singularity.

1313. Prove that if $E \in N_F$, then E is F-removable in any domain $G \supset E$ (removablencss of nul sets).

HINT. Make use of the result of the preceding problem.

1314. Prove that if $E_k \in N_F$ $(k = 1, 2, \ldots, q)$ and $E_i \cap E_j = 0$ $(i \neq j)$, then $E = \bigcup_{k=1}^{q} E_k \subset N_F$ (the propcrty of analyticity of nul sets).

1315. Prove that if $E \subset N_F$, then E does not contain interior points.

HINT. Assuming that E contains the circle $|z-a| \leqslant \varrho$, consider the function

$$f(z) = \begin{cases} a + \dfrac{\varrho^2}{z-a}, & \text{if } |z-a| > \varrho, \\ a + \bar{z} - \bar{a}, & \text{if } |z-a| \leqslant \varrho. \end{cases}$$

1316. Prove that if $E \in N_F$, then the plane measure $mE = 0$ (a stronger form of the result of problem **1313**).

HINT. On the assumption that E is bounded (this is not essential) and $mE > 0$, consider the function (see problem **1058**)

$$f(z) = \iint_E \frac{d\xi\, d\eta}{\zeta - z} \qquad (\zeta = \xi + i\eta).$$

1317. Prove that if E forms a rectifiable arc, then $E \in N_{AC}$.

HINT. Make use of the generalised theorem of Cauchy (if $f(z)$ is continuous in the closed region $\overline{G}$, bounded by a rectifiable contour Γ, and is analytic inside G, then $\int_\Gamma f(z)\,dz = 0$) and Morera's theorem.

A set E is said to be everywhere discontinuous, if in an arbitrarily small neighbourhood of every point $z \in E$ it is possible to draw a closed Jordan curve γ, $\gamma \cap E = 0$, surrounding the peint z. A bounded, perfeet, everywhere discontinuous set can be included in a finite number of closed rectifiable Jordan curves $\{\gamma_k\}$ $(k = 1, 2, \ldots, q)$ with diameters less than an arbitrarily specified number $\varepsilon > 0$, lying outside one another and not having points in eommon with E. The quantity

$$l(E) = \inf \sum_{k=1}^{q} l(\gamma_k),$$

where $l(\gamma_k)$ is the length of γ_k and inf is applied to all possible systems $\{\gamma_k\}$, is known as the length $l(E)$ of the set E.

1318. Prove the following assertions:

(1) If E is a perfect everywhere discontinuous set of finite length $l(E) < \infty$, then $E \in N_{AC}$.

HINT. For the domain $G \supset E$ and $\varepsilon > 0$ there exists $d(\varepsilon) > 0$ such that for z', $z'' \in G$, $|z'-z''| < d$ we have $|f(z')-f(z'')| < \varepsilon$. For the system of contours $\{\gamma_k\}$ with diameters less than d, containing E and having the sum of lengths $\sum_k l(\gamma_k) < l(E)+1$, the expansion

$$f(z) = \phi(z) + \sum_k \psi_k(z)$$

of problem **1312** is constructed and by means of the relations (ζ_k is inside γ_k, z is in G outside $\{\gamma_k\}$),

$$\psi_k(z) = \frac{1}{2\pi i}\int_{\gamma_k}\frac{f(\zeta)\,d\zeta}{\zeta - z} = \frac{1}{2\pi i}\int_{\gamma_k}\frac{f(\zeta)-f(\zeta_k)}{\zeta - z}\,d\zeta$$

$\left|\sum_k \psi_k(z)\right|$ is estimated and this leads to the equality $f(z) = \phi(z)$.

(2) If a perfect set E is situated on a rectifiable arc, then $E \in N_F$.

(3) There exist bounded, perfect, everywhere discontinuous sets $E \in N_{AC}$ which have infinite length.

(4) There exist bounded, perfect, everywhere discontinuous sets E of infinite length and plane measure zero, $l(E) = \infty$, $mE = 0$, not belonging to N_{AC}.

HINT. See V. V. GOLUBEV, *Single-valued analytic functions with perfect sets of singular points, (Odnoznachnyye analiticheskiye funktsii s sovershennym mnozhestvom osobykh tochek)*, Saratov, 1916.

1319. Prove the following assertions on the class of null sets N_{AB}.

(1) If the set E contains a continuum, then $E \overline{\in} N_{AB}$.

(2) $N_{AB} \subset N_{AC}$, correspondingly $O_{AC} \subset O_{AB}$ (the inclusion is strict).

(3) If E is a perfect everywhere discontinuous set and the length $l(E) = 0$, then $E \in N_{AB}$.

HINT. As above (see the preceding hint) use the expansion $f(z) = \phi(z) + \sum_k \psi_k(z)$, but with $\sum_k l(\gamma_k) < \varepsilon$ and estimate $\left|\sum_k \psi_k(z)\right|$ for the purpose of proving the equality $f(z) = \phi(z)$.

(4) If E is a perfect set, situated on a straight line, then in order that $E \in N_{AB}$, it is necessary and sufficient that the linear measure mes $E = 0$.

HINT. Assuming that mes $E > 0$, consider the function

$$f(z) = \int_E \frac{dt}{t - z}.$$

1320. Prove that if $E \in N_{AD}$, then every function $f(z)$, which is schlicht and conformally maps the domain Ω which is the complement of E, is linear.

HINT. Use the AD-removableness of E. To prove that $f(z)$ is schlicht on E use the principle of corresponding boundaries, after enclosing E by some contour.

CHAPTER X

CONFORMAL MAPPINGS (CONTINUATION)

§ 1. The Schwarz–Christoffel formula†

Let us denote by P a polygon in the w-plane, by $A_k (k = 1, 2, \ldots, n)$ its vertices, arranged in the order of positive traversal of P with respect to its interior, and by $\alpha_k \pi$ its internal angles $\left(\sum_{k=1}^{n} \alpha_k = n-2\right)$. The function $w = f(z)$ which maps the upper half-plane $\operatorname{Im} z > 0$ onto the interior of the polygon P, is determined by the Schwarz–Christoffel formula:

$$f(z) = C \int_0^z \prod_{k=1}^{n} (z-a_k)^{\alpha_k - 1} \, dz + C_1, \tag{1}$$

where $-\infty < a_1 < a_2 < \ldots < a_n < \infty$ are points on the x-axis which correspond to the vertices $A_1, A_2, \ldots, A_n$ of the polygon P; C and C_1 are complex constants.

The mapping of the upper half-plane onto the exterior of the same polygon is effected by the function

$$F(z) = C \int_0^z \prod_{k=1}^{n} (z-a'_k)^{\beta_k - 1} \frac{dz}{(z-b)^2 (z-\bar{b})^2} + C_1, \tag{2}$$

where b is the point of the upper half-plane corresponding to the point at infinity of the w-plane; a'_k are points which correspond to the vertices A_k of the polygon (now $\infty > a'_1 > a'_2 > \ldots > a'_k > -\infty$); the $\beta_k \pi$ are the external angles of the polygon $(\beta_k = 2-\alpha_k, \sum_{k=1}^{n} \beta_k = n+2)$. If it is a question of the mapping of the exterior of a polygon some of the points A_k may be geometrically coincident.

1321. Prove that in formula (1), of the n points a_k, three can be assigned arbitrarily, and the remaining $n-3$ points and the con-

† For this section see [1, Chapter VIII, § 7], [3, Chapter II, § 3]; B. A. FUCHS and B. V. SHABAT, *Functions of a Complex Variable*, (*Funkisii kompleksnogo peremennogo*), 2nd ed., Chap. VIII, Fizmatgiz, 1959. English translation published by Pergamon Press (1962).

stants C and C_1 (in all $n+1$ real parameters) are determined from the relations

$$\int_{a_i}^{a_{i+1}} |f'(x)|\,dx = |C| \int_{a_i}^{a_{i+1}} \prod_{k=1}^{n} |x-a_k|^{\alpha_k-1} dx = l_i$$

$$\left(i = 1, 2, \ldots, n;\ a_{n+1} = a_1,\ \text{and} \int_{a_n}^{a_1} = \int_{a_n}^{\infty} + \int_{-\infty}^{a_1}\right),$$

where the l_i are the lengths of the sides A_iA_{i+1} ($A_{n+1} = A_1$), $C_1 = f(a_1) = w_0$ (w_0 is the affix of the vertex A_1) and $\arg C = \theta$ (θ is the angle of inclination of the side A_nA_1 of the polygon to the real axis).

The given relations exceed the unknown parameters in number, however: they determine these parameters uniquely. Explain this and show which of the relations can be discarded.

1322. Prove that if one of the vertices of the polygon is the image of the point at infinity, for example $a_n = \infty$, then the mapping function is of the form

$$f(z) = C \int_0^z \prod_{k=1}^{n-1} (z-a_k)^{\alpha_k-1} dz + C_1.$$

HINT. Make the transformation $\zeta = -1/z$, if none of the a_k equals 0 and $\zeta = -\dfrac{1}{z-a}$, where $a \neq a_k$, if one of the points $a_k = 0$ ($k = 1, 2, \ldots, n-1$).

1323. Prove that if one or several of the vertices A_k is at ∞, then formula (1) remains in force, if by $\alpha_k\pi$ is understood the angle between the corresponding rays at the finite point of their intersection, taken with a negative sign.

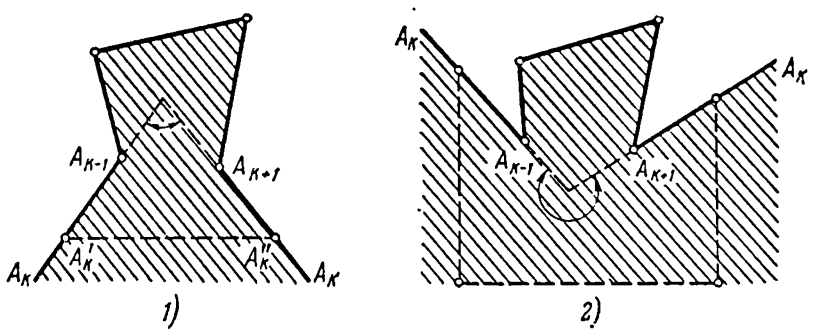

FIG. 32

HINT. Let $A_k = \infty$. If $\alpha_k < 1$, then consider the polygon P', cut off from P by the segment $A'_k A''_k$ where A'_k and A''_k lie sufficiently far from the sides $A_{k-1}A_k$ and $A_{k+1}A_k$ (Fig. 32 (1)) and in formula (1) for P' pass to the limit as $A'_k \to \infty$, $A''_k \to \infty$. If, however, $\alpha_k \geqslant 1$, then join $A'_k A''_k$ in P by a polygonal line (see Fig. 32(2)) and by enlarging it similarly extend it to infinity.

1324. Determine the quantities α_k, which enter into formula (1) for the infinitely distant vertex of the "polygon" formed from parallel rays represented in Fig. 33.

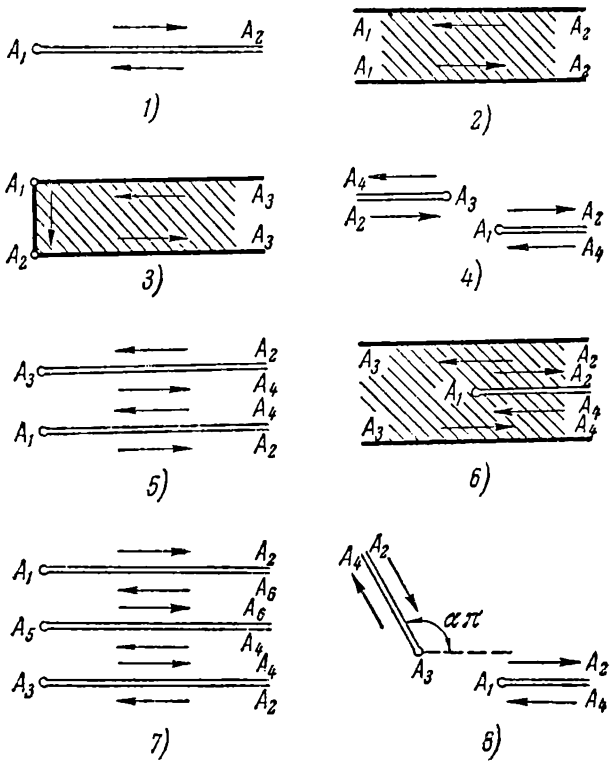

FIG. 33

HINT. Pass to the limit as recommended in the hint to problem **1323**.

1325. (1) Prove that in the mapping of the unit circle $|z| < 1$ onto the polygon P, situated in the finite part of the plane, the mapping function is of the form

$$f(z) = C \int_0^z \prod_{k=1}^{n} (z - a_k)^{\alpha_k - 1} \, dz + C_1,$$

where $a_k = e^{i\phi_k}(\phi_1 < \phi_2 < \ldots < \phi_n)$ are the points on the circle $|z| = 1$, corresponding to the vertices A_k, traversed in the positive direction and the $\alpha_k\pi$ are the internal angles of the polygon P.

(2) Prove that the function which maps the unit disk $|z| < 1$ onto the exterior of the same polygon P, subject to the condition that the point $z = 0$ passes into the point $w = \infty$, is of the form

$$f(z) = C\int_1^z \prod_{k=1}^{n} (z-a_k')^{\beta_k - 1}\frac{dz}{z^2},$$

where $a_k' = e^{i\phi_k'}$ $(\phi_1' > \phi_2' > \ldots > \phi_k')$, and the $\beta_k\pi$ are the external angles of P.

1326. Find all the cases of single-valued transformation of the Schwarz–Christoffel formula (1), that is, explain for which polygons P the inverse function $z = z(w)$ is defined and single valued in the whole w-plane.

HINT. Polygons obtained from P by any number of reflections in the sides, must without omissions or overlapping cover the whole w-plane.

In problems **1327–1332** map the upper half-plane $\operatorname{Im} z > 0$ onto the given domain P, located in the w-plane with the given correspondence of the vertices of P and the points of the real axis. Determine also the period or periods of the inverse function $z(w)$, the group G of its invariant linear transformations of the w-plane and the fundamental domain B of this group (see page 30).

1327. (1) P is the strip $0 < v < h$; $w(-\infty, \infty) \to z\ (0, \infty)$†.

(2) P is the strip $0 < v < h$; $w(-\infty, \infty) \to z(-1, 1)$.

1328. P is the half-strip $0 < u < \pi$; $v < 0$; $w(0, \pi, -i\infty) \to z$ $(1, -1, \infty)$.

1329. (1) P is the right-angled triangle with acute angles $\pi/3$, $\pi/6$; $w\left(0, \omega, \omega + \frac{i\omega}{\sqrt{3}}\right) \to z(0, 1, \infty)$.

(2) P is the right-angled isosceles triangle;

$$w(0, \omega, \omega + i\omega) \to z(0, 1, \infty).$$

(3) P is the equilateral triangle;

$$w\left(0, \omega, \omega\frac{1+i\sqrt{(3)}}{2}\right) \to z(0, 1, \infty).$$

† Here and in what follows by the symbol $w(A, B, \ldots) \to z(a, b, \ldots)$ is meant the correspondence of points of the w- and z- planes $A \leftarrow\!\!\to a$, $B \leftarrow\!\!\to b, \ldots$

1330. Find the domain of the w-plane onto which the function

$$w(z) = \int_0^z t^{\alpha-1}(1-t)^{\beta-1}\,dt \qquad (\arg w'(\tfrac{1}{2}) = 0)$$

maps the upper half-plane $\operatorname{Im} z > 0$, if:

(1) $0 < \alpha \leqslant 2$, $0 < \beta \leqslant 2$; consider the cases: (a) $\alpha+\beta < 1$, (b) $\alpha+\beta = 1$, (c) $\alpha+\beta > 1$, in particular $\alpha+\beta = 2$ and $\alpha = \beta = 3/2$

(2) $1 \leqslant \alpha \leqslant 2$, $-1 \leqslant \beta \leqslant 0$, $\alpha+\beta \geqslant 1$; consider the cases: (a) $\alpha = 1$, (b) $\alpha+\beta = 1$, (c) $\alpha = 2$, (d) $\alpha = 2$, $\beta = -\frac{1}{2}$, (e) $\alpha = 2$, $\beta = -1$.

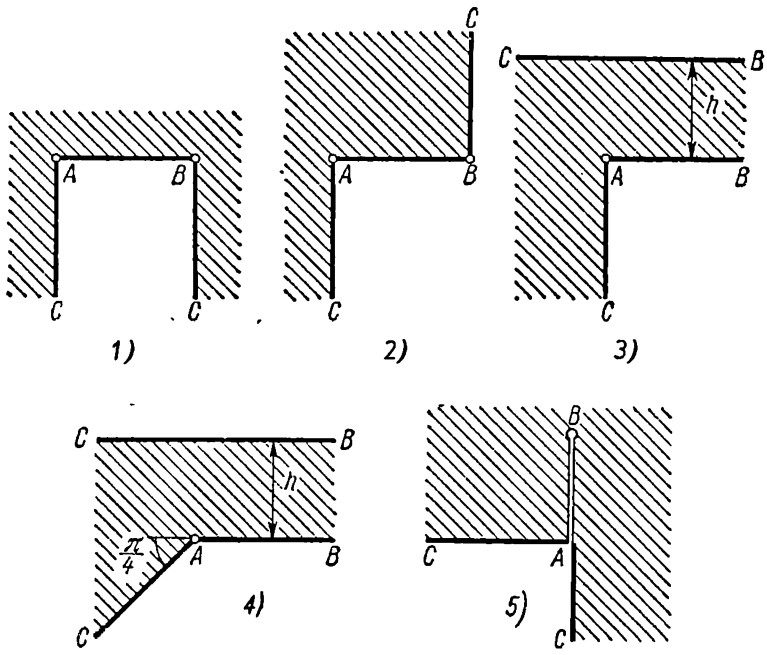

FIG. 34

1331. (1) Find the domains of the w-plane onto which the function

$$w(z) = \int_0^z \frac{t-\lambda}{\sqrt{[t(t-1)]}}\,dt$$

$$= \sqrt{[z(z-1)]} + (1-2\lambda)\left(\log\left[\sqrt{(z)} - \sqrt{(z-1)}\right] - \tfrac{1}{2}i\pi\right)$$

maps the upper half-plane. Consider the cases: $\lambda < 0$, $0 < \lambda < \frac{1}{2}$, $\lambda = \frac{1}{2}$, $\frac{1}{2} < \lambda < 1$ and $\lambda > 1$.

(2) Find the domains of the w-plane onto which the function

$$w(z) = \frac{1}{2\pi i}\int_1^z \frac{1-\lambda}{\sqrt{[t(t-1)]}}\frac{dt}{t} = \frac{1}{\pi i}\left[\log\left[\sqrt{(z)}+\sqrt{(z-1)}\right]-\lambda\sqrt{\left(\frac{z-1}{z}\right)}\right]$$

maps the upper half-plane. Consider the cases: $\lambda < 0$, $0 < \lambda < 1$ and $\lambda > 1$.

1332. Map the upper half-plane $\operatorname{Im} z > 0$ onto the domains of the w-plane indicated in Fig. 34 with the given correspondence of points.

(1) $w\,(A = 0,\ B = 1,\ C = \infty) \to z\,(0, 1, \infty)$;
(2) $w\,(A = 0,\ B = 1,\ C = \infty) \to z\,(0, 1, \infty)$;
(3) $w\,(A = 0,\ B = \infty,\ C = \infty) \to z\,(0, 1, \infty)$;
(4) $w\,(A = 0,\ B = \infty,\ C = \infty) \to z\,(0, 1, \infty)$;
(5) $w\,(A = 0,\ B = ia,\ C = \infty) \to z\,(0, 1, \infty)$.

1333. Map the upper half-plane $\operatorname{Im} z > 0$ onto the domains in the w-plane indicated in Fig. 35 $(0 < \theta < 1)$.

(1) $w\,(A,\ B = 0,\ C = \infty) \to z\,(0, 1, \infty)$.
(2) $w\,(A,\ B = 0,\ C = \infty) \to z\,(0, 1, \infty)$.

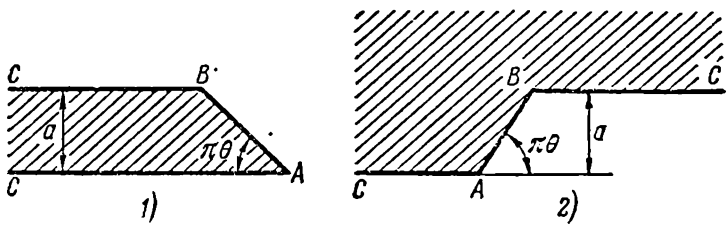

FIG. 35

In the case when θ is a rational number $(\theta = p/q)$, express the integrals obtained in terms of elementary functions.

1334. Map the upper half-plane onto the domain indicated in Fig. 36 (the arc AC is a semicircle).

$$w(A = a_i,\ B = \infty,\ C = 0) \to z\,(0, 1, \infty).$$

HINT. This is reduced to a particular case of problem **1333**, (2) by means of the mapping $\zeta = a/w$.

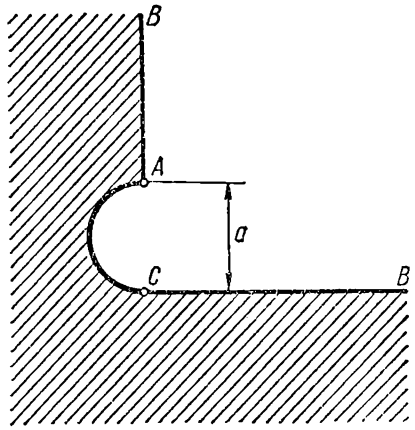

FIG. 36

1335. Map the upper half-plane $\operatorname{Im} z > 0$ onto the domain of the w-plane indicated in Fig. 37, with the condition

$$w(A = -h,\ B = \infty,\ C = h,\ D = \infty) \to z\ (-1, 0, 1, \infty).$$

(The possibility of this mapping follows from the symmetry principle.)

1336. Map the upper half-plane $\operatorname{Im} z > 0$ onto the lozenge in the w-plane with angle $\pi\alpha$ at the vertex A and side d (Fig. 38). The corresponding points are given by

$$w(A = 0,\ B = d,\ C = d(1+e^{i\pi\alpha}),\ D = de^{i\pi\alpha}) \to z\ (0,\ 1,\ \infty, -1).$$

Justify the possibility of this mapping.

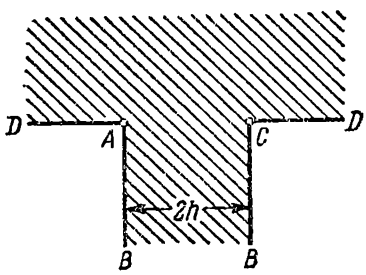

FIG. 37

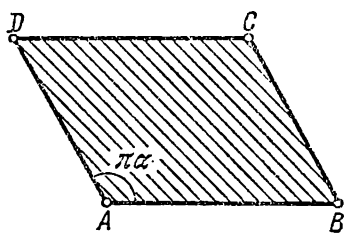

FIG. 38

1337. Map the upper half-plane $\operatorname{Im} z > 0$ onto the domain in the w-plane indicated in Fig. 39. The parameters a, b ($a > 0$,

$b > 0$) are the images of the corresponding vertices, they cannot be specified arbitrarily and they must be found.

(1) $w\ (A = 0, B = \infty, C = H+ih, D = \infty) \to z\ (0, 1, a, \infty)$.

(2) $w\ (B = \infty, C = H+ih, D = \infty, C' = -H+ih, B' = \infty) \to z\ (1, a, \infty, -a, -1)$.

HINT. Make an additional cut along the imaginary axis and use the symmetry principle.

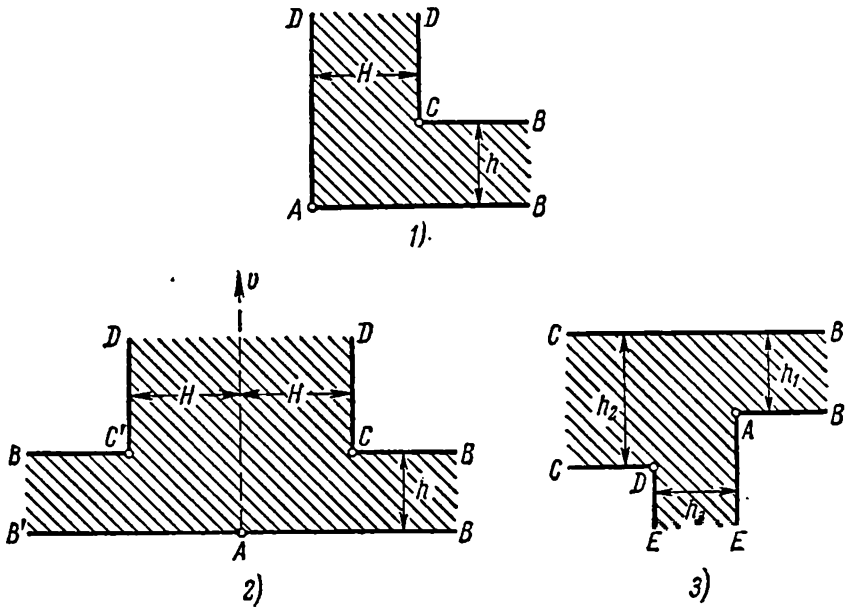

FIG. 39

(3) $w\ (A = 0, \quad B = \infty, \quad C = \infty, \quad D = -h_3 - i(h_2 - h_1), E = \infty) \to z\ (0, 1, a, \infty, -b)$.

1338. (1) Find the function $w(z)$ which maps the disk $|z| < 1$ onto the interior of the regular n-gon with centre at the coordinate origin and one of its vertices at the point $w = 1$, subject to the conditions that $w\ (0) = 0$, $w'(0) > 0$.

(2) Map the disk $|z| < 1$ onto the exterior of the same n-gon subject to the conditions that $w(0) = \infty$, $w(x) > 0$ $(0 < x < 1)$.

Determine c_{-1} in the expansion $w(z) = \dfrac{c_{-1}}{z} + \ldots$.

1339. (1) Find the function $w(z)$ which maps the disk $|z| < 1$ onto the interior of the regular five pointed star with centre at

the coordinate origin and one of the vertices at the point $w = 1$; the condition of normalisation: $w(0) = 0$, $w'(0) > 0$.

(2) Map the disk $|z| < 1$ onto the exterior of the same star; $w(0) = \infty$, $w(x) > 0$ $(0 < x < 1)$; determine c_{-1} in the expansion

$$w(z) = \frac{c_{-1}}{z} + \ldots .$$

1340. Find the domain onto which the function

$$w(z) = \int_0^z \frac{(1+t^n)^\lambda \, dt}{(1-t^n)^{\frac{2}{n}+\lambda}}, \quad w'(0) > 0, \quad -1 < \lambda < 1 - \frac{2}{n}$$

maps the unit disk $|z| < 1$.

1341. Map the upper half-plane $\operatorname{Im} z > 0$ onto the regular n-gon in the w-plane with centre at the coordinate origin and one of the vertices at the point $w = 1$; the normalisation is $w(i) = 0$, $w(0) = 1$.

1342. Let the domain P in the w-plane be the exterior of the "star" consisting of n segments issuing from the point $w = 0$ (Fig. 40). Let A_k be the vertices of P at the coordinate origin and $\alpha_k \pi$ the corresponding angles $(\sum_{k=1}^{n} \alpha_k = 2)$, B_k the vertices of P at the ends of the segments of the star $(A_1, B_1, A_2, B_2, \ldots$ are arranged

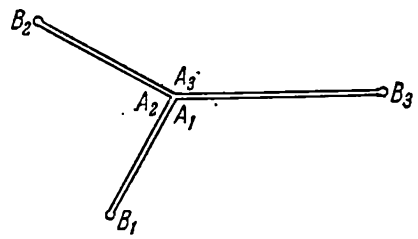

FIG. 40

in the order of positive traversal of P), $l_k = \overline{A_k B_k}$ the lengths of the segments of the star. Prove that the function $w = f(z)$, $f(0) = \infty$ which maps the unit disk $|z| < 1$ onto P is of the form

$$f(z) = \frac{C}{z} \prod_{k=1}^{n} (z - a_k)^{\alpha_k},$$

where C is a complex constant and $a_k = e^{i\phi_k}$ are the points on the circle $|z| = 1$ which correspond to the A_k. The points $b_k = e^{i\psi_k}$ which correspond to the vertices B_k are the roots of the equation

$$\sum_{k=1}^{n} \frac{\alpha_k}{z-a_k} - \frac{1}{z} = 0. \tag{3}$$

How are the parameters C, a_k, b_k determined?

HINT. In continuation by the symmetry principle the function $f(z)$ is multiplied by a constant factor, hence the function

$$\phi(z) = \frac{f'(z)}{f(z)}$$

is single valued in the z-plane.

REMARK. This formula immediately gives the solution of a number of problems of Chapter II, for example, problems 281, (2), 282(1), 284, 290.

It is recommended that these problems should be solved again and the constants entering into the general formula determined.

1343. Map the exterior of the double angle (Fig. 41) onto the exterior of the circle $|z| > 1$ subject to the conditions: $w(\infty) = \infty$, $w'(\infty) > 0$.

1344. Prove that the function $w = f(z)$, which maps the upper half-plane $\operatorname{Im} z > 0$ onto the exterior of the star of problem **1342** is of the form

$$f(z) = C \frac{\prod_{k=1}^{n} (z-a_k)^{\alpha_k}}{(z-z_0)(z-\bar{z}_0)},$$

where z_0 is the point of the upper half-plane which is transformed into ∞.

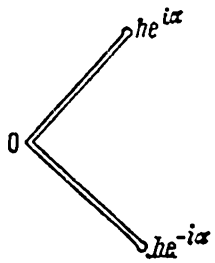

FIG. 41

1345. Let the domain P in the w-plane be bounded by the ray $[0, \infty)$, by $n-1$ segments, going from $w = 0$ to the points B_i

($i = 1, 2, \ldots, n-1$), and $m-1$ rays going from the points D_s ($s = 1, 2, \ldots, m-1$) to ∞, such that their continuations in the opposite direction pass through the coordinate origin (Fig. 42). Let A_k ($k = 1, 2, \ldots, n$) be the vertices of P at the origin, $\alpha_k \pi$ the cor-

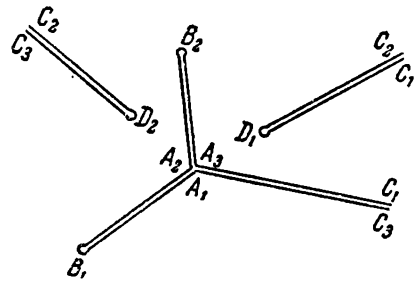

FIG. 42

responding angles, $C_j (j = 1, 2, \ldots, m)$ the vertices of P at ∞, $\gamma_j \pi$ the corresponding angles, $A_1 B_1 A_2 \ldots A_n C_1 D_1 \ldots C_m$ a positive traversal of the boundary of P. Prove that the function $w = f(z)$, which maps the upper half-plane $\operatorname{Im} z > 0$ onto P, is of the form

$$f(z) = C \frac{\prod_{k=1}^{n} (z - a_k)^{\alpha_k}}{\prod_{j=1}^{m} (z - c_j)^{\gamma_j}}, \tag{4}$$

where a_k, c_j are the points on the x-axis which correspond to the vertices A_k, C_j. The points b_i, d_s on the x-axis which correspond to the vertices B_i, D_s are the roots of the equation

$$\sum_{k=1}^{n} \frac{\alpha_k}{z - a_k} - \sum_{j=1}^{m} \frac{\gamma_j}{z - c_j} = 0. \tag{5}$$

How are the parameters C, a_k, b_i, c_j, d_s determined? What is changed if one of the parameters a_k, b_i, c_j, d_s is equal to ∞?

HINT. Prove that

$$\frac{f'(z)}{f(z)} = \sum_{k=1}^{n} \frac{\alpha_k}{z - a_k} - \sum_{j=1}^{m} \frac{\gamma_j}{z - c_j} = \frac{\prod_{i=1}^{n-1} (z - b_i) \prod_{s=1}^{m-1} (z - d_s)}{\prod_{k=1}^{n} (z - a_k) \prod_{i=1}^{m} (z - c_j)}.$$

1346. Prove that formula (4) of problem **1345** still applies to the case where the domain P, contains two rays $[0, \infty)$ (Fig. 43). Here the vertex at the coordinate origin is taken into account also in the case when there issue from it only two rays forming a single straight line.

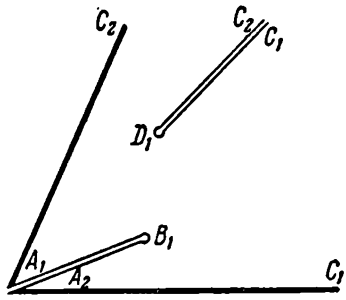

FIG. 43

In problems **1347–1351** map onto the upper half-plane $\operatorname{Im} z > 0$ the domains of the w-plane given in the corresponding diagrams subject to the given conditions, and determine the parameters a and b $(a > 0,\ b > 0)$.

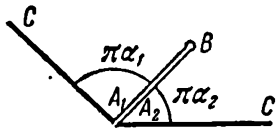

FIG. 44

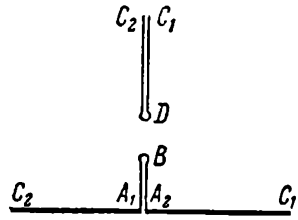

FIG. 45

1347. The domain indicated in Fig. 44,

$$w\,(A_1 = 0,\ B = he^{i\pi\alpha_1},\ A_2,\ C = \infty) \to z\,(-1, b, 1, \infty).$$

1348. The domain indicated in Fig. 45,

$$w\,(A_1 = 0,\ B = ih,\ A_2,\ C_1 = \infty,\ D = iH,\ C_2 = \infty) \to z\left(-a, 0, a, \frac{1}{a}, \infty, -\frac{1}{a}\right).$$

1349. The domain indicated in Fig. 46,

$$w\,(A_1 = 0,\ B_1 = -he^{i\pi\alpha},\ A_2,\ B_2 = he^{i\pi\alpha},\ A_3,\ C = \infty) \to z\,(-1,\ -b,\ 0,\ b,\ 1,\ \infty).$$

1350. The domain indicated in Fig. 47,

$$w\,(A_1 = 0,\ B = he^{i\pi\alpha_1},\ A_2,\ C_1 = \infty,\ D = He^{i\pi\alpha_2},\ C_2 = \infty) \to z\left(-a,\ 0, b, \frac{1}{b},\ \infty,\ -\frac{1}{a}\right).$$

FIG. 46

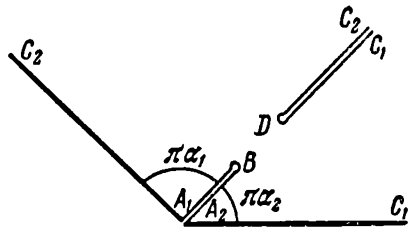

FIG. 47

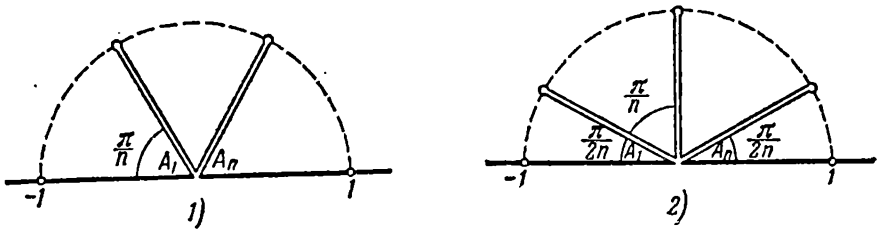

FIG. 48

1351. (1) The domain indicated in Fig. 48, (1),

$$w\,(\infty) = \infty,\ w\,(\pm 1) = \pm 1.$$

(2) The domain indicated in Fig. 48, (2) (the angles between the cuts are π/n; the extreme segments form with the corresponding rays of the real axis angles $\pi/2n$),

$$w(\infty) = \infty, \qquad w(A_1, A_n) \to z\,(-1, 1).$$

1352. Let the domain P in the w-plane be the horizontal strip of width H with cuts going left to ∞ from the points B_i $(i = 1,$

$2, \ldots, n-1$) and right from the points D_s ($s = 1, 2, \ldots, m-1$) (Fig. 49). Prove that the function $w = f(z)$, which maps the upper half-plane $\operatorname{Im} z > 0$ onto the domain P is of the form

$$f(z) = \sum_{k=1}^{n} \frac{h_k}{\pi} \log (z - a_k) - \sum_{j=1}^{m} \frac{l_j}{\pi} \log (z - c_j) + C, \qquad (6)$$

where a_k, c_j are the points on the x-axis corresponding to the vertices A_k and C_j of the domain P; the h_k are the distances between the cuts on the left, and the l_j are the corresponding distances on the right. The points $b_i (i = 1, 2, \ldots, n-1)$ and d_s ($s = 1, 2, \ldots, m-1$) on the x-axis corresponding to the vertices B_i and D_s are the roots of the equation

$$\sum_{k=1}^{n} \frac{h_k}{z - a_k} - \sum_{j=1}^{m} \frac{l_j}{z - c_j} = 0. \qquad (7)$$

How are the parameters C, a_k, b_i, c_j, d_s determined? What is changed if one of the parameters a_k, b_i, c_j, d_s becomes equal to ∞?

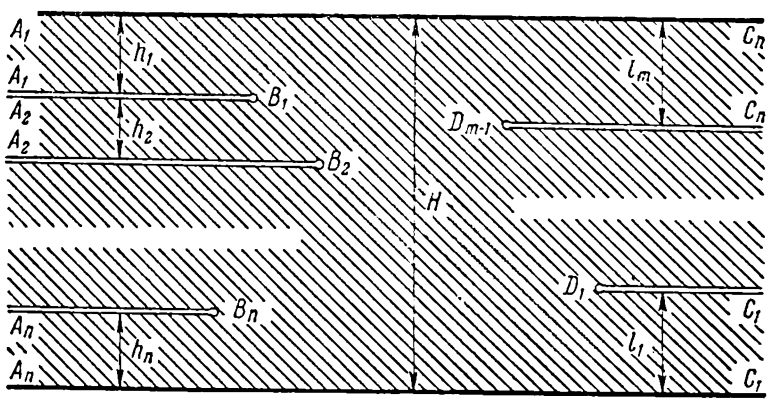

FIG. 49

HINT. The problem reduces to problem **1346**. It is also possible to find the function $f'(z)$ which is single valued or to derive directly from the Schwarz–Christoffel formula.

1353. Let the domain P be the upper half-plane $\operatorname{Im} w > 0$ with horizontal cuts going leftwards to ∞ from the points B_i ($i = 1, 2, \ldots, n$) and to ∞ on the right from the points D_s ($s = 1, 2, \ldots, m$) (Fig. 50).

Prove that the function $w = f(z)$ which maps the upper half-plane $\operatorname{Im} z > 0$ onto the domain P is of the form

$$f(z) = \sum_{k=1}^{n} \frac{h_k}{\pi} \log (z - a_k) - \sum_{j=1}^{m} \frac{l_j}{\pi} \log (z - c_j) + Az + B, \qquad (8)$$

where a_k, c_j are points on the x-axis corresponding to the vertices A_k and C_j of the strips of the domain P, and h_k, l_j are the distances

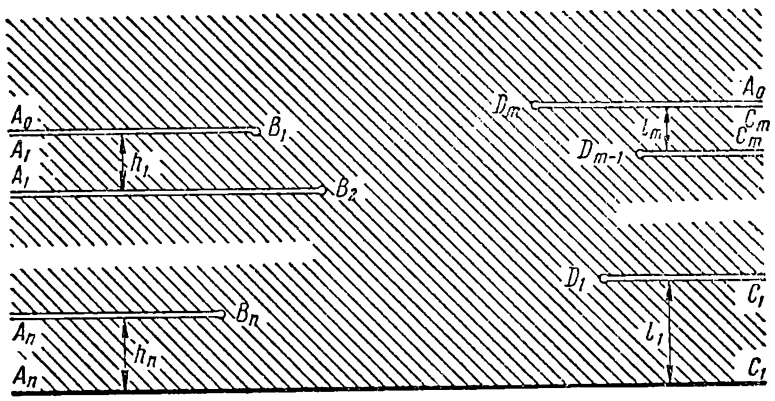

FIG. 50

between the cuts going in one direction. The point A_0 is transformed into ∞, $A > 0$ and $\operatorname{Im} B = \sum_{j=1}^{m} l_j$. The points $b_i (i = 1, 2, \ldots, n)$ and d_s $(s = 1, 2, \ldots, m)$ on the x-axis corresponding to the vertices B_i and D_s are roots of the equation

$$\sum_{k=1}^{n} \frac{h_k}{z - a_k} - \sum_{j=1}^{m} \frac{l_j}{z - c_j} + A\pi = 0. \qquad (9)$$

How are the parameters A, B, a_k, b_i, c_j, d_s determined? Show that if A_0 passes into a point $a_0 \neq \infty$, then

$$f(z) = \sum_{k=1}^{n} \frac{h_k}{\pi} \log (z - a_k) - \sum_{j=1}^{m} \frac{l_j}{\pi} \log (z - c_j) + \frac{L - H}{\pi} \log (z - a_0) + \frac{A}{z - a_0} + B, \qquad (10)$$

where

$$H = \sum_{k=1}^{n} h_k, \quad L = \sum_{j=1}^{m} l_j, \quad A < 0 \text{ and } \operatorname{Im} B = L.$$

If the parameter a_k or c_j is equal to ∞, then in formula (10) the corresponding term is absent.

HINT. Use the Schwarz-Christoffel formula. To determine the coefficients of the logarithmic terms compare the increments of w on going round the points a_0, a_k, c_j (along a semicircle) calculated geometrically and by the formula for $f(z)$.

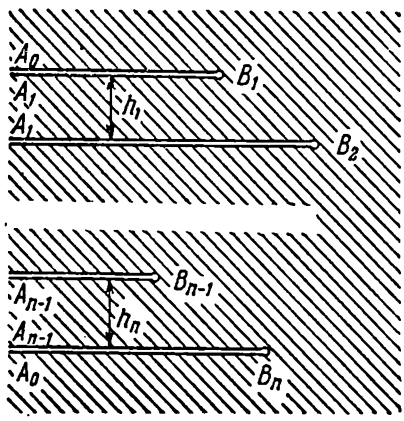

FIG. 51

1354. Let the domain P be the w-plane with horizontal cuts going from the points $B_i (i = 1, 2, \ldots, n)$ to ∞ on the left (Fig. 51). Prove that the function $w = f(z)$, which maps the upper half-plane $\operatorname{Im} z > 0$ onto the domain P, is of the form

$$f(z) = -Az^2 + Bz + C + \sum_{k=1}^{n-1} \frac{h_k}{\pi} \log (z - a_k),$$

where the a_k are points on the x-axis corresponding to the vertices A_k of the strips of the domain P, and h_k are the distances between the cuts. The point A_0 maps into ∞, $A > 0$ and B is a real number. The points $b_i (i = 1, 2, \ldots, n)$ corresponding to the vertices B_i are zeros of the derivative $f'(z)$. How are the parameters A, B, C, a_k, b_i determined?

HINT. See the hint to problem **1353**.

1355. Let the domain P be the w-plane with horizontal cuts going to ∞ on the left and on the right (Fig. 52). Prove that the function

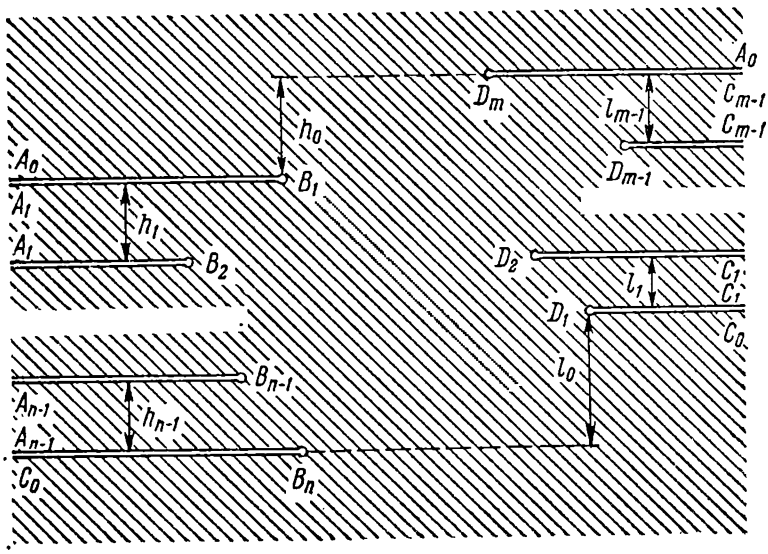

FIG. 52

$w = f(z)$, which maps the upper half-plane $\operatorname{Im} z > 0$ onto the domain P, is of the form

$$f(z) = \sum_{k=0}^{n-1} \frac{h_k}{\pi} \log (z-a_k) - \sum_{j=0}^{m-1} \frac{l_j}{\pi} \log (z-c_j) + \frac{A}{z-a_0} + \frac{C}{z-c_0} + B, \tag{11}$$

where $A > 0$, $C > 0$, $h_0 = \operatorname{Im}(D_m - B_1)$, $l_0 = \operatorname{Re}(D_1 - B_n)$, and the remaining parameters (including b_i and d_s for the vertices B_i and D_s) have the same values as in problem **1353**. How are the parameters A, B, C, a_k, b_i, c_j, d_s determined? Show that if $a_0 = \infty$, then

$$f(z) = \sum_{k=1}^{n-1} \frac{h_k}{\pi} \log (z-a_k) - \sum_{j=0}^{m-1} \frac{l_j}{\pi} \log (z-c_j) - \frac{C}{z-c_0} + Az + B, \tag{12}$$

where $A > 0$ and $C > 0$. If the parameter $a_k (k \neq 0)$ or $c_j (j \neq 0)$ is equal to ∞, then in formula (12) the corresponding term is absent.

HINT. See the hint to problem **1353**.

1356. Map the upper half-plane $\operatorname{Im} z > 0$ onto the domains in the w-plane indicated in Fig. 53 (all the dimensions are given on the corresponding diagrams) for the given conditions; find a and b $(a > 0,\ b > 0)$.

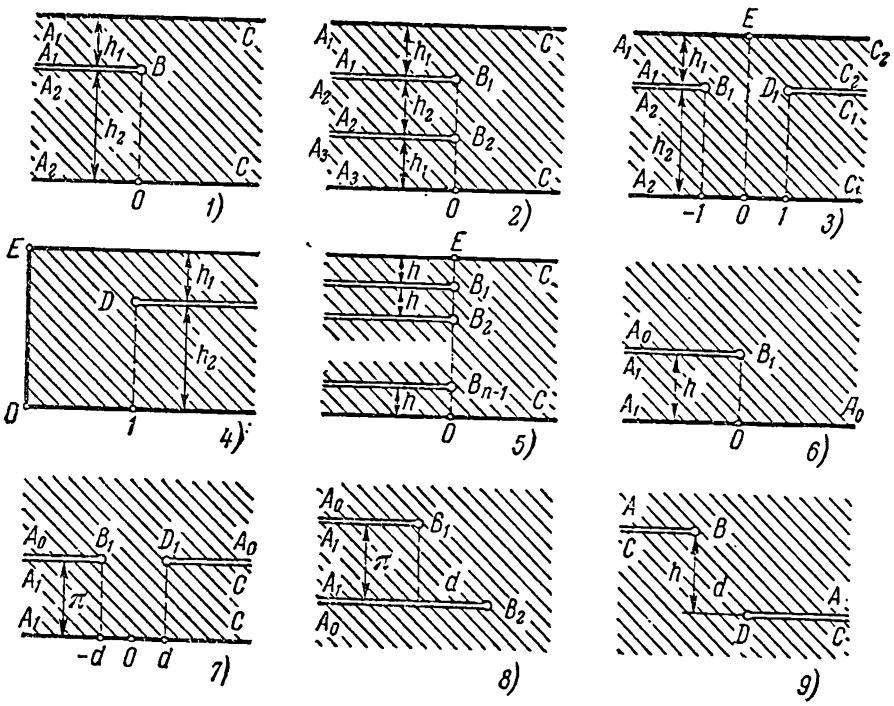

FIG. 53

(1) $(A_1, B, A_2, C) \to (-1, b, 1, \infty)$;
(2) $(A_1, B_1, A_2, B_2, A_3, C) \to (-1, -b, 0, b, 1, \infty)$;
(3) $(E, O, D,) \to (-1, 1, \infty)$;
(4) $(E, O, D) \to (-1, 1, \infty)$;
(5) $(E, O, C) \to (-1, 1, \infty)$;
(6) $(A_0, B, A_1) \to (\infty, 0, 1)$;
(7) $(A_1, O, C) \to (-1, 0, 1)$;
(8) $(A_0, B_1, A_1, B_2) \to (\infty, -(1 + a), -1, 0)$; $d = \operatorname{Re}(B_2 - B_1)$;
(9) $(A, B, C, D) \to (\infty, -1, a, 1)$; $d = \operatorname{Re}(D - B)$; $h = \operatorname{Im}(B - D)$.

§ 2. Conformal mappings involving the use of elliptic functions†

The integral

$$u = \int_0^z \frac{dz}{\sqrt{[(1-z^2)(1-k^2z^2)]}} = \int_0^\phi \frac{d\phi}{\sqrt{(1-k^2\sin^2\phi)}} = F(\phi, k) \tag{1}$$

is known as the *normal elliptic integral of the first kind in Legendre's form.* The inverse function

$$z = \operatorname{sn}(u, k) \tag{2}$$

is one of the fundamental *elliptic functions of Jacobi*, known as the sn *function of Jacobi.* Two other elliptic functions of Jacobi are connected with it:

$$\operatorname{cn}(u, k) = \sqrt{[1-\operatorname{sn}^2(u, k)]}, \quad \operatorname{dn}(u, k) = \sqrt{[1-k^2\operatorname{sn}^2(u, k)]}, \tag{3}$$

called respectively the cn- and dn-*functions of Jacobi.* If there is no necessity to indicate the modulus k, then they are simply written $\operatorname{sn} u$, $\operatorname{cn} u$ and $\operatorname{dn} u$. It follows from (1)–(3), that

$$\begin{aligned} \frac{d\operatorname{sn} u}{du} &= \operatorname{cn} u \operatorname{dn} u, \\ \frac{d\operatorname{cn} u}{du} &= -\operatorname{sn} u \operatorname{dn} u, \\ \frac{d\operatorname{dn} u}{du} &= -k^2 \operatorname{sn} u \operatorname{cn} u. \end{aligned} \tag{4}$$

In what follows, if nothing is said to the contrary, it is assumed that $0 < k < 1$.

The integrals

$$\begin{aligned} K &= \int_0^1 \frac{dz}{\sqrt{[(1-z^2)(1-k^2z^2)]}} = F\left(\frac{\pi}{2}, k\right), \\ K' &= \int_0^1 \frac{dz}{\sqrt{[(1-z^2)(1-k'^2z^2)]}} = F\left(\frac{\pi}{2}, k'\right) \end{aligned} \tag{5}$$

are known as *complete elliptic integrals of the first kind*, and $k' = \sqrt{(1-k^2)}$ as the *complementary parameter.*

† For this seetion see: N. I. Akhiyezer, *Elements of the Theory of Elliptic Functions* (*Elementy teorii ellipticheskikh funktsii*), Gostekhizdat, 1948; Yu. S. Sikorskii, *Elements of the Theory of Elliptic Functions with Applications to Mechanics* (*Elementy teorii ellipticheskikh funktsii s prilozheniyami k mekhanike*), Gostekhizdat, 1936; A. M. Zhuravskii, *Textbook on Elliptic Functions*, (*Spravochnik po ellipticheskim funktsiyam*), Gostekhizdat, 1941; A. Betz, *Konforme Abbildung*, Berlin, 1948.

1357. Investigate the mapping of the z-plane and the ϕ-plane by means of the normal elliptic integral of the first kind (1). Continuing this mapping by the symmetry principle, prove that sn u is a doubly periodic function with periods $4K$ and $2iK'$.

HINT. Consider the mapping of the upper half-plane Im $z > 0$ by means of the principle of corresponding boundaries. Trace the variation of the argument of du/dz when z is varied along the x-axis.

1358. Map the upper half-plane Im $z > 0$ onto the rectangle in the w-plane with vertices $\pm a$, $\pm a \pm ib$ $(a > 0,\ b > 0)$.

1359. Investigate the mapping of the z-plane by means of the elliptic integral

$$w = \int_z^1 \frac{dz}{\sqrt{[(1-z^2)(k'^2+k^2z^2)]}}.$$

Find the inverse function and prove that it is doubly periodic with periods $4K$, $2K + 2iK'$.

HINT. Make the substitution $z = \sqrt{(1-t^2)}$.

1360. Investigate the mapping of the z-plane by means of the elliptic integral

$$w = \int_z^1 \frac{dz}{\sqrt{[(1-z^2)(z^2-k'^2)]}}.$$

Find the inverse function and prove that it is doubly periodic with periods $2K$, $4iK'$.

HINT. Make the substitution $z = \sqrt{(1-k^2t^2)}$.

1361. Verify Table 1 and Table 2 (apart from the last column). The dash $(')$ denotes passage to the complementary parameter k'.

HINT. Table 2 is obtained from Table 1. Let R be the rectangle with vertices $(0, K, K+iK', iK')$ in the u-plane and R_1 its image in the u_1-plane. To establish the relationship between $z_1 = \text{sn}\,(u_1, k)$ and $z = \text{sn}\,(u, k)$ map the image R onto the image R_1, preserving the correspondence of vertices. This applies to all the rows of Table 1 except the last where as R there is taken the rectangle with vertices $(0, K', K'+iK, iK)$ and a correspondence is established between z_1 and $z' = \text{sn}'\,u = \text{sn}\,(u, k')$.

1362. Consider the mapping of the z_1-plane onto the u_1-plane by means of the elliptic integral of the first kind $u_1 = F(\phi_1, k_1)$ $\left((z_1 = \sin\phi_1 = \text{sn}\,(u_1, k_1)\right)$ for the following values of k_1: $\frac{1}{k}$, k', $\frac{1}{k'}$,

TABLE 1

u_1	$\operatorname{sn}(u_1, k)$	$\operatorname{cn}(u_1, k)$	$\operatorname{dn}(u_1, k)$
$-u$	$-\operatorname{sn} u$	$\operatorname{cn} u$	$\operatorname{dn} u$
$u+2K$	$-\operatorname{sn} u$	$-\operatorname{cn} u$	$\operatorname{dn} u$
$u+2iK'$	$\operatorname{sn} u$	$-\operatorname{cn} u$	$-\operatorname{dn} u$
$u+K$	$\dfrac{\operatorname{cn} u}{\operatorname{dn} u}$	$-k'\dfrac{\operatorname{sn} u}{\operatorname{dn} u}$	$k'\dfrac{1}{\operatorname{dn} u}$
$u+iK'$	$\dfrac{1}{k \operatorname{sn} u}$	$-\dfrac{i \operatorname{dn} u}{k \operatorname{sn} u}$	$-i\dfrac{\operatorname{cn} u}{\operatorname{sn} u}$
iu	$i\dfrac{\operatorname{sn}' u}{\operatorname{cn}' u}$	$\dfrac{1}{\operatorname{cn}' u}$	$\dfrac{\operatorname{dn}' u}{\operatorname{cn}' u}$

TABLE 2

u	$\operatorname{sn} u$	$\operatorname{cn} u$	$\operatorname{dn} u$	$Z(u)+\dfrac{\operatorname{dn} u \operatorname{cn} u}{\operatorname{sn} u}$
$K+i\eta$	$\dfrac{1}{\operatorname{dn}' \eta}$	$-\dfrac{ik' \operatorname{sn}' \eta}{\operatorname{dn}' \eta}$	$\dfrac{k' \operatorname{cn}' \eta}{\operatorname{dn}' \eta}$	$-i\left\{Z'(\eta)+\dfrac{\eta\pi}{2KK'}\right\}$
$-K+i\eta$	$-\dfrac{1}{\operatorname{dn}' \eta}$	$\dfrac{ik' \operatorname{sn}' \eta}{\operatorname{dn}' \eta}$	$\dfrac{k' \operatorname{cn}' \eta}{\operatorname{dn}' \eta}$	$-i\left\{Z'(\eta)+\dfrac{\eta\pi}{2KK'}\right\}$
$\xi+iK'$	$\dfrac{1}{k \operatorname{sn} \xi}$	$-\dfrac{i \operatorname{dn} \xi}{k \operatorname{sn} \xi}$	$-i\dfrac{\operatorname{cn} \xi}{\operatorname{sn} \xi}$	$Z(\xi)-\dfrac{i\pi}{2K}$
$\xi-iK'$	$\dfrac{1}{k \operatorname{sn} \xi}$	$\dfrac{i \operatorname{dn} \xi}{k \operatorname{sn} \xi}$	$\dfrac{i \operatorname{cn} \xi}{\operatorname{sn} \xi}$	$Z(\xi)+\dfrac{i\pi}{2K}$
$i\eta$	$\dfrac{i \operatorname{sn}' \eta}{\operatorname{cn}' \eta}$	$\dfrac{1}{\operatorname{cn}' \eta}$	$\dfrac{\operatorname{dn}' \eta}{\operatorname{cn}' \eta}$	$-i\left\{Z'(\eta)+\dfrac{\operatorname{dn}' \eta \operatorname{cn}' \eta}{\operatorname{sn}' \eta}+\dfrac{\pi\eta}{2KK'}\right\}$
$\pm iK'$	$\dfrac{1}{k}$	$\mp\dfrac{i}{k}$	$\mp i$	—
	Residues			

$\frac{ik}{k'}$, $\frac{ik'}{k}$, and verify Table 3, where the notation $K_1 = K(k_1)$, $K_1' = K(k_1')$ $(k_1' = \sqrt{(1-k_1^2)})$ is used.

HINT. To establish the connection between z_1 and $z = \operatorname{sn}(u, k)$ compare the mappings of the first quadrants of the z_1-plane and the z-plane by means of the functions $F(\phi_1, k_1)$ and $F(\phi, k)$ (in the case of $k_1 = k'$ take the second quadrant of the z_1-plane). The necessary relationships are given in the third column.

TABLE 3

k_1	u_1	$\operatorname{sn}(u_1, k_1)$	$\operatorname{cn}(u_1, k_1)$	$\operatorname{dn}(u_1, k_1)$	K_1	K_1'
$\frac{1}{k}$	ku	$k \operatorname{sn}(u, k)$	$\operatorname{dn}(u, k)$	$\operatorname{cn}(u, k)$	$k(K+iK')$	kK'
k'	iu	$i\frac{\operatorname{sn}(u, k)}{\operatorname{cn}(u, k)}$	$\frac{1}{\operatorname{cn}(u, k)}$	$\frac{\operatorname{dn}(u, k)}{\operatorname{cn}(u, k)}$	K'	K
$\frac{1}{k'}$	$ik'u$	$ik'\frac{\operatorname{sn}(u, k)}{\operatorname{cn}(u, k)}$	$\frac{\operatorname{dn}(u, k)}{\operatorname{cn}(u, k)}$	$\frac{1}{\operatorname{cn}(u, k)}$	$k'(K'+iK)$	$k'K$
$\frac{ik}{k'}$	$k'u$	$k'\frac{\operatorname{sn}(u, k)}{\operatorname{dn}(u, k)}$	$\frac{\operatorname{cn}(u, k)}{\operatorname{dn}(u, k)}$	$\frac{1}{\operatorname{dn}(u, k)}$	$k'K$	$k'(K'+iK)$
$\frac{ik'}{k}$	iku	$ik\frac{\operatorname{sn}(u, k)}{\operatorname{dn}(u, k)}$	$\frac{1}{\operatorname{dn}(u, k)}$	$\frac{\operatorname{cn}(u, k)}{\operatorname{dn}(u, k)}$	kK'	$k(K+iK')$

The integral

$$\int_0^z \sqrt{\left(\frac{1-k^2z^2}{1-z^2}\right)}\,dz = \int_0^u \sqrt{(1-k^2\sin^2\phi)}\,d\phi = E(\phi, k) \tag{6}$$

is known as the *normal elliptic integral of the second kind*. By the substitution $z = \operatorname{sn} u$ it is reduced to the form

$$\int_0^z \sqrt{\left(\frac{1-k^2z^2}{1-z^2}\right)}\,dz = \int_0^u \operatorname{dn}^2 u\,du = E(u). \tag{7}$$

Two functions of Jacobi, $Z(u)$ and $\theta(u)$ are connected with this integral:

$$Z(u) = E(u) - \frac{E}{K}u = \frac{\theta'(u)}{\theta(u)}, \tag{8}$$

where

$$E = E\left(\frac{\pi}{2}, k\right) = \int_0^1 \sqrt{\left(\frac{1-k^2z^2}{1-z^2}\right)}\,dz \tag{9}$$

is the *complete elliptic integral of the second kind.* The quantities K, E and K' E', obtained by passing to the complementary parameter k', are connected by the relations

$$EK'+E'K-KK' = \pi/2. \tag{10}$$

We shall also use the notation

$$D = \frac{K-E}{k^2} = \int_0^1 \frac{z^2\,dz}{\sqrt{[(1-z^2)(1-k^2z^2)]}}\,. \tag{11}$$

Making the substitution $k^2z^2+k'^2t^2 = 1$, it is possible to establish the formulae

$$\int_1^z \frac{dz}{\sqrt{[(1-z^2)(1-k^2z^2)]}} = i\left(K' - \int_0^t \frac{dt}{\sqrt{[(1-t^2)(1-k'^2t^2)]}}\right), \tag{12}$$

$$\int_1^z \sqrt{\left(\frac{1-k^2z^2}{1-z^2}\right)}\,dz = i(K'-E') + i\left(\int_0^t \sqrt{\left(\frac{1-k'^2t^2}{1-t^2}\right)}dt - \int_0^t \frac{dt}{\sqrt{[(1-t^2)(1-k'^2t^2]}}\right). \tag{13}$$

Making use of the notation for the elliptic integrals given by (1) and (7) these formulae can be written briefly as:

$$u-K = i(K'-u_1),$$

$$E(u)-E = i(K'-E')+i[E'(u_1)-u_1],$$

where

$$u_1 = \int_0^t \frac{dt}{\sqrt{[(1-t^2)(1-k'^2t^2)]}}\,.$$

In particular, it follows from formula (13) that

$$\int_1^{\frac{1}{k}} \sqrt{\left(\frac{1-k^2z^2}{1-z^2}\right)}\,dz = i(K'-E')\,.$$

The integral

$$\int_0^z \frac{dz}{(1+\nu z^2)\sqrt{[(1-z^2)(1-k^2z^2)]}} = \int_0^\phi \frac{d\phi}{(1+\nu\sin^2\phi)\sqrt{(1-k^2\sin^2\phi)}} = \Pi(\phi, \nu, k) \qquad (14)$$

is called the *normal elliptic integral of the third kind.* By the substitution $z = \operatorname{sn} u$ it is reduced to the form

$$\int_0^u \frac{du}{1+\nu\operatorname{sn}^2 u}, \qquad (15)$$

and is expressed in terms of the θ-function of Jacobi. The quantity

$$\Pi_1 = \Pi_1(\nu, k) = \Pi\left(\frac{\pi}{2}, \nu, k\right) = \int_0^1 \frac{dz}{(1+\nu z^2)\sqrt{[(1-z^2)(1-k^2z^2)]}} \qquad (16)$$

is known as the *complete elliptic integral of the third kind.* The quantity Π_1 can be expressed in terms of elliptic integrals of the first and second kinds†.

1363. Consider the mapping of the z_1-plane onto the w_1-plane by means of the elliptic integral of the second kind $w_1 = E(\phi_1, k_1)$ $(z_1 = \sin\phi_1 = \operatorname{sn} u_1)$ for the following values of the parameter $k_1 : \frac{1}{k}, k', \frac{1}{k'}, \frac{ik}{k'}, \frac{k'}{ik}$ $(0 < k < 1)$ and verify Table 4 (page 235).

HINT. See the hint to problem **1362**.

1364*. Map the upper half-plane $\operatorname{Im} z > 0$ onto the upper half-plane $\operatorname{Im} w > 0$ with two vertical cuts along the segments $\operatorname{Re} w = \pm a, 0 < \operatorname{Im} w < h$ with the normalisation: $w(0) = 0, w(\infty) = \infty$, $w'(\infty) > 0$.

HINT, Use the Schwarz–Christoffel formula in the form

$$w(z) = Ck^2 \int_0^z \frac{b^2 - z^2}{\sqrt{[(1-z^2)(1-k^2z^2)]}}\, dz = C\left[\int_0^z \sqrt{\left(\frac{1-k^2z^2}{1-z^2}\right)}\, dz - (1-k^2b^2)\int_0^z \frac{dz}{\sqrt{[(1-z^2)(1-k^2z^2)]}}\right];$$

† For the calculation of elliptic integrals of the third kind and of related quantities see ERDELYI, MAGNUS, OBERHETTINGER, TRICOMI, (1953), *Higher, Transcendental Functions*, vol. II. Note that for $\nu \in (-\infty, -1), (-1, -k^2), (-k^2, 0), (0, \infty)$ the formulae for the calculation of (15) and (16) have different forms.

TABLE 4

k'	u_1	$E(u_1, k_1)$	E_1	E_1'
$1/k$	ku	$\frac{1}{k}[E(u)-k'^2u]$	$\frac{1}{k}[(E-k'^2K)-i(E'-k^2K')]$	$(1/k)E'$
k'	iu	$i\left[u-E(u)+\frac{\operatorname{sn} u \operatorname{dn} u}{\operatorname{cn} u}\right]$	E'	E
$1/k'$	$ik'u$	$\frac{i}{k'}\left[k'^2u-E(u)+\frac{\operatorname{sn} u \operatorname{dn} u}{\operatorname{cn} u}\right]$	$\frac{1}{k'}[(E'-k^2K')-i(E-k'^2K)]$	$(1/k')E$
$\frac{ik}{k'}$	$k'u$	$\frac{1}{k'}\left[E(u)-k^2\frac{\operatorname{sn} u \operatorname{cn} u}{\operatorname{dn} u}\right]$	$(1/k')E$	$\frac{1}{k'}[(E'-k^2K')-i(E-k'^2K')]$
$\frac{ik'}{k}$	iku	$\frac{1}{k}\left[E'(u)-k'^2\frac{\operatorname{sn}' u \operatorname{cn}' u}{\operatorname{dn}' u}\right]$	$(1/k)E'$	$\frac{1}{k}[(E-k'^2K)-i(E'-k^2K')]$

for the determination of the constants use the values of a and h with the equation $w\left(\frac{1}{k}\right) - w(1) = 0$ (it determines b in terms of k) and the condition $w'(\infty) > 0$.

1365. Verify the last column of Table 2, (see problem **1361**) and, using it, consider the mapping of the rectangle R: $|\xi| < K$, $|\eta| < K'$, in the u-plane ($u = \xi + i\eta$), by means of the function

$$w = Z(u) + \frac{\operatorname{dn} u \operatorname{cn} u}{\operatorname{sn} u}.$$

HINT. Use the equations

$$k'^2 \int_0^u \frac{dt}{\operatorname{cn}^2 t} = k'^2 u - E(u) + \frac{\operatorname{sn} u \operatorname{dn} u}{\operatorname{cn} u},$$

$$k'^2 \int_0^u \frac{dt}{\operatorname{dn}^2 t} = E(u) - k^2 \frac{\operatorname{sn} u \operatorname{cn} u}{\operatorname{dn} u},$$

easily verified by differentiation (see formulae (4)).

1366. Investigate the mapping of the z-plane by means of the normal elliptic integral of the third kind (14) for $0 < k < 1$. Consider separately the cases when ν belongs to the intervals:

(1) $(-\infty, -1)$; (2) $(-1, -k^2)$; (3) $(-k^2, 0)$; (4) $(0, \infty)$. Consider also the cases: $\nu = -1$ and $\nu = -k^2$.

HINT. Investigate the mapping of the first quadrant of the z-plane.

The integral

$$w = \int_z^\infty \frac{dz}{\sqrt{[4z^3 - g_2 z - g_3]}} = \int_z^\infty \frac{dz}{\sqrt{[4(z-e_1)(z-e_2)(z-e_3)]}} \tag{17}$$

with discriminant $\Delta = g_2^3 - 27 g_3^2 \neq 0$ (with this condition no two of e_1, e_2, e_3 are equal) is known as the *normal elliptic integral of the first kind in Weierstrass's form*, and the function

$$z = \mathfrak{p}(w) \tag{18}$$

is known as the *"p"-function of Weierstrass*. This is one of the fundamental elliptic functions with periods 2ω, $2\omega'$ ($\operatorname{Im} \frac{\omega'}{\omega} \neq 0$). It satisfies the differential equation

$$\mathfrak{p}'^2 = 4\mathfrak{p}^3 - g_2 \mathfrak{p} - g_3 = 4(\mathfrak{p} - e_1)(\mathfrak{p} - e_2)(\mathfrak{p} - e_3). \tag{19}$$

The function $\mathfrak{p}(w)$ is even, two-sheeted in the parallelogram of periods (Fig. 54), having there a pole of the second order at zero and the double points $(\mathfrak{p}' = 0)\,\omega,\ \omega+\omega',\ \omega'$:

$$e_1 = \mathfrak{p}(\omega),\quad e_2 = \mathfrak{p}(\omega+\omega'),\quad e_3 = \mathfrak{p}(\omega'). \tag{20}$$

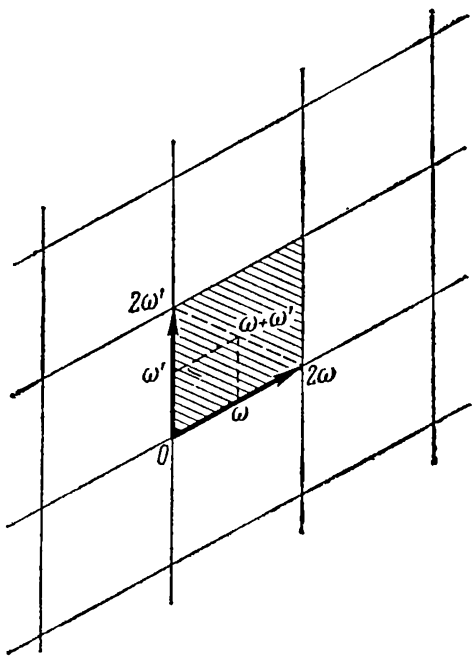

FIG. 54

It follows from (19) that

$$\left.\begin{aligned} e_1+e_2+e_3 &= 0,\\ e_1e_2+e_2e_3+e_3e_1 &= -\frac{1}{4}g_2,\\ e_1e_2e_3 &= \frac{1}{4}g_3. \end{aligned}\right\} \tag{21}$$

1367*. Investigate the mapping of the z-plane by means of the normal elliptic integral of the first kind in Weierstrass's form (17) for real g_2, g_3 and $\Delta > 0$. Consider the cases $g_3 > 0$, $g_3 < 0$, $g_3 = 0$. Find the periods of $\mathfrak{p}(w)$.

HINT. Consider the mapping of the upper half-plane $\operatorname{Im} z > 0$ by the principle of corresponding boundaries.

1368*. Investigate the mapping of the z-plane by means of the normal elliptic integral of the first kind (17) for real g_2, g_3 and $\Delta < 0$. In particular, consider the case $g_2 = 0$. Find the periods of $\mathfrak{p}(w)$.

HINT. Since $\Delta < 0$, two of the quantities e_1, e_2, e_3, are complex conjugates, and one is real. Let e_2 be the real quantity, $e_1 = \alpha + i\beta$, $e_3 = \alpha - i\beta$ $(\beta > 0)$. Consider the mapping of the semicircle $|z - e_2| = |e_1 - e_2|$, $\operatorname{Im} z > 0$ by means of the principle of corresponding boundaries and continue this mapping by the symmetry principle.

1369. Find the mapping onto the upper half-plane $\operatorname{Im} w > 0$ of the triangle ABC with the given conditions:

(1) $(A = 0,\ B = \omega > 0,\ C = \omega(1+i)) \to (\infty, -1, 0)$;

(2) $\left(A = 0,\ B = a > 0,\ C = \frac{a\sqrt{3}}{2} e^{\frac{i\pi}{3}}\right) \to (\infty, -1, 1)$;

(3) $\left(A = 0,\ B = a > 0,\ C = \frac{a\sqrt{3}}{2} e^{\frac{i\pi}{3}}\right) \to (\infty, -1, 0)$.

HINT. Use the solutions of problem **1367** (the case $g_3 = 0$) and **1368** (the case $g_2 = 0$).

REMARK. For this problem see also problems **1330–1332.**

1370. Map the doubly connected domains 1–15 of the z-plane shown in Fig. 55 onto the circular ring $\varrho_1 < |w| < \varrho_2$ and determine the modulus $\mu = \varrho_2/\varrho_1$ (see page 29).

In problems **1371–1373** map the given domains onto the unit circle $|t| < 1$.

1371. The rectangle $\operatorname{Re}|u| < K$, $\operatorname{Im}|u| < K'$ $(0 < k < 1)$. Find the positions of the vertices in the mapping.

1372. The interior of the ellipse $|z - 1| + |z + 1| = 2a$ $(a > 1)$ with the cuts $[-a, -1]$, $[1, a]$.

1373. The interior of the ellipse $|z - 1| + |z + 1| = 2a$ $(a > 1)$. Find the positions of the foci in the mapping.

1374. Map the exterior of the unit circle $|t| > 1$ onto the domains 1 — 3, of the z-plane indicated in Fig. 56.

1375†. Let R be the rectangle $|\xi| < K$, $|\eta| < K'$ in the plane of $u = \xi + i\eta$. Using Table 2 (see problem **1361**) and the principle of corresponding boundaries, prove that the functions

$$\frac{z}{l} = \frac{\operatorname{dn} u}{\operatorname{sn} u - f - ig \operatorname{cn} u},$$

$$\frac{z}{l} = Z(u) + \frac{\operatorname{dn} u\,(\operatorname{cn} u + ig \operatorname{sn} u + ih)}{\operatorname{sn} u - f - ig \operatorname{cn} u} + mu$$

(all the parameters are real, $f^2 + g^2 < 1$, $m > 0$, $0 < k < 1$) map the rectangle R onto the exterior of a cross, the dimensions of which

† For this problem see DARWIN, (1950), Some conformal transformations involving elliptic functions, *Phil. Mag.*, 7, vol. 41, No. 312, 1–11.

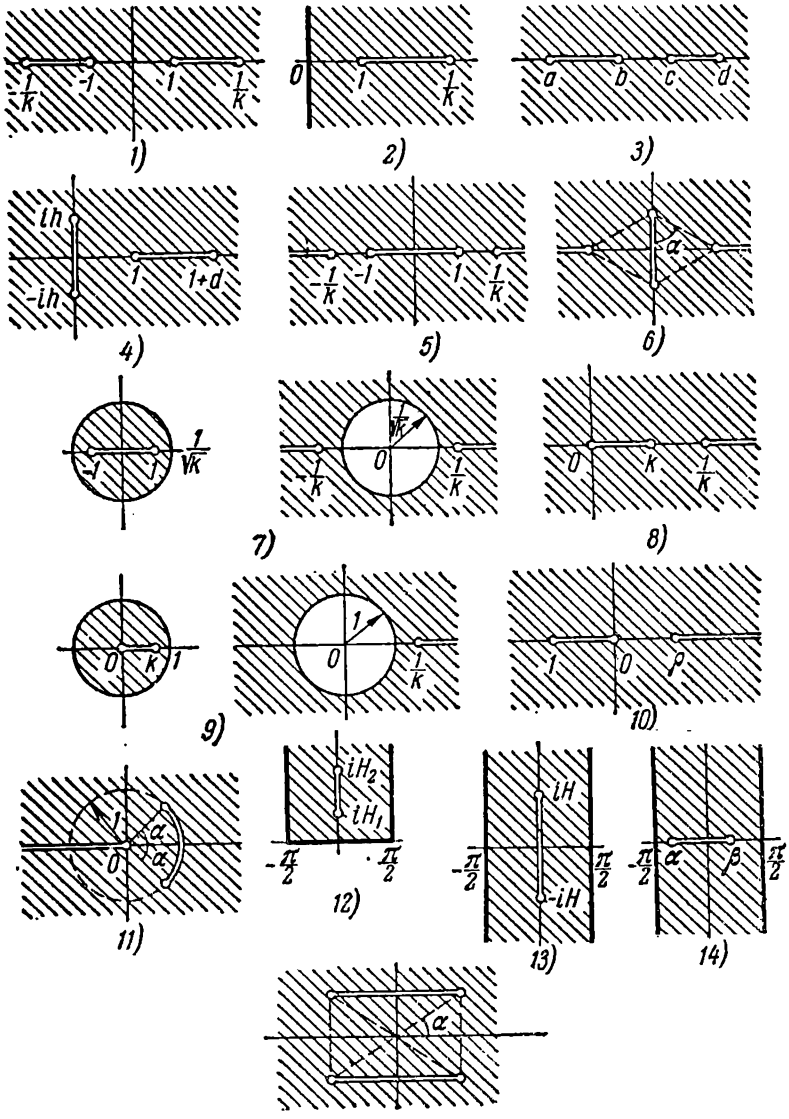

FIG. 55

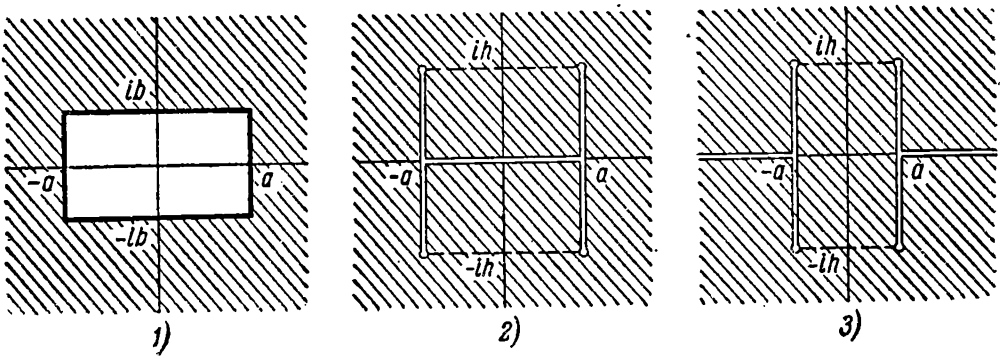

FIG. 56

are determined by the parameters l, f, g, k, or onto the exterior of a rectangle with four branches—continuations of the sides, the dimensions of which are determined by the parameters l, f, g, h, m, k. In particular, the functions

(1) $\operatorname{dn} u/\operatorname{sn} u$,

(2) $Z(u) + \operatorname{dn} u \operatorname{cn} u/\operatorname{sn} u$,

(3) $Z(u) + \dfrac{\operatorname{dn} u \operatorname{cn} u}{\operatorname{sn} u} + mu$,

(4) $Z(u) + \dfrac{\operatorname{dn} u \operatorname{cn} u}{\operatorname{sn} u} + \dfrac{ih \operatorname{dn} u}{\operatorname{sn} u}$,

(5) $Z(u) + \dfrac{\operatorname{dn} u \operatorname{cn} u}{\operatorname{sn} u - f}$,

(6) $Z(u) + \dfrac{\operatorname{dn} u \operatorname{cn} u}{\operatorname{sn} u} + \dfrac{ih \operatorname{dn} u}{\operatorname{sn} u} + mu$,

(7) $Z(u) + \dfrac{\operatorname{dn} u \operatorname{cn} u}{\operatorname{sn} u} + \dfrac{ih \operatorname{dn} u}{\operatorname{sn} u} + \dfrac{\pi}{4K^2} \quad \left(k = \dfrac{1}{\sqrt{2}}\right)$

map R onto the exterior of the diagrams 1–7 in Fig. 57.

REMARK. Each of the functions (1)–(7) corresponds to the figures of a horizontal row of Fig. 57. The figures of one such row illustrate the effect (qualitatively) of the variation of some parameter, namely:

In the cases (1), (2) the parameter k increases (with motion to the right).

In case (3) the parameter m increases. For $m = \dfrac{E}{K} - k'^2$ a rectangle is obtained, in particular, if $k = \dfrac{1}{\sqrt{2}}$ $\left(\text{then } m = \dfrac{\pi}{4K^2}\right)$, it is a square. The last figure of the series is obtained for $m = \dfrac{\pi}{2KK'}$.

In the case (4) the parameter h increases from zero. For $h = \dfrac{1}{kk'}\left(\dfrac{E}{K} - k'^2\right)$ one pair of branches disappears.

In case (5) the parameter f increases. For $f = \dfrac{1}{k}\left(1 - k'^2 \dfrac{K}{E}\right)$ one pair of branches disappears.

Case (6) differs from case (4) by the presence of the term with m, which causes thickening.

Case (7) is the particular case of (6) when $k = 1/\sqrt{2}$, $m = \pi/4K^2$; a "windmill" is obtained.

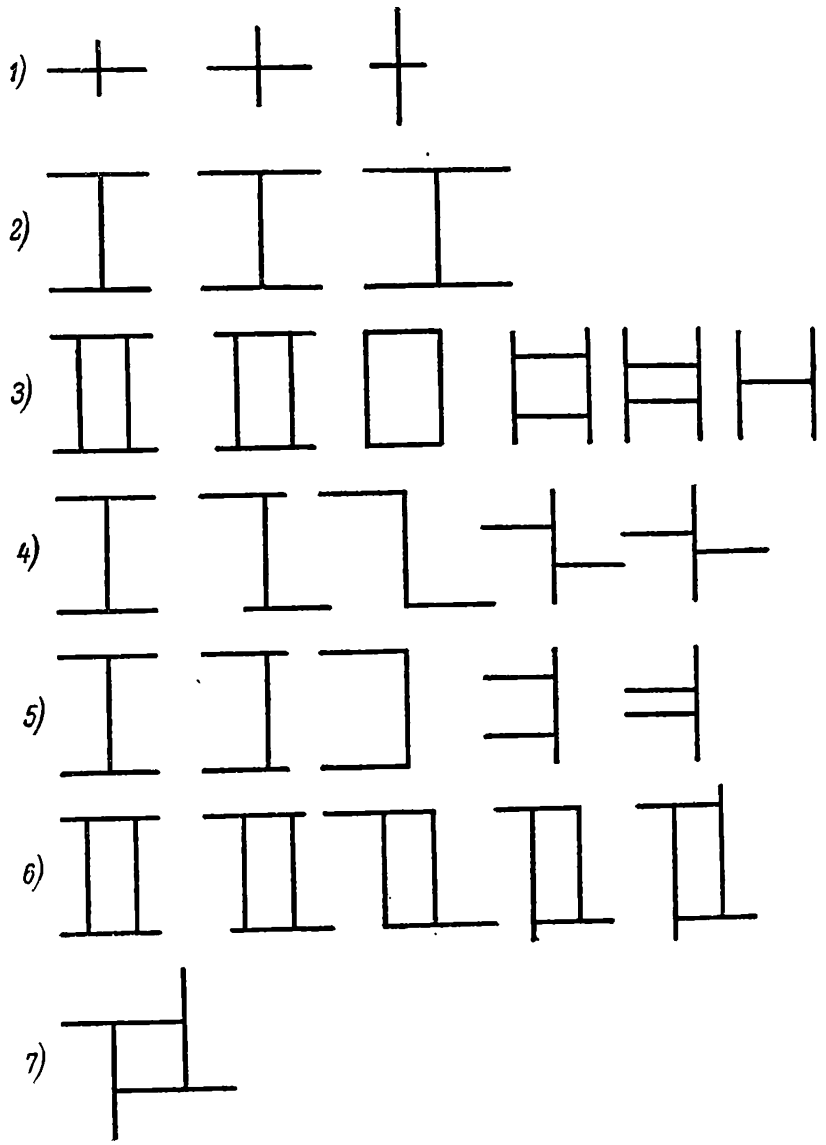

FIG. 57

On the application of elliptic functions to the problems of the mapping of the upper half-plane onto the exterior of an ellipse, hyperbola and parabola, and also on the mapping of the exterior of two arbitrarily situated rectilinear segments or of two concentric arcs onto a circular ring see, for example, L. I. SEDOV, *Plane Problems of Hydrodynamics and Aerodynamics*, (*Ploskiye zadachi gidrodinamiki i aerodinamiki*), Chapter VI, § 2, Gostekhizdat, 1950; W. KOPPENFELS, F. STALLMAN, *Praxis der konformen Abbildung*, Berlin, 1959 (in this book there is also a catalogue of mappings of different types of polygons).

CHAPTER XI

APPLICATIONS TO MECHANICS AND PHYSICS†

§ 1. Applications to hydrodynamics:

Steady flows

The established plane irrotational flow of an incompressible fluid is described by the analytic function

$$w(z) = \phi(x, y) + i\psi(x, y) \tag{1}$$

known as the *complex potential* or *characteristic function* of the flow; ϕ is known as the *potential function*, ψ as the *stream function*. The curves $\phi = \text{const}$ are the *equipotential lines*, the curves $\psi = \text{const}$ are the *stream lines*. The stream velocity $\mathbf{V}$ is connected with $w(z)$ by the relations:

$$\left.\begin{aligned} &\mathbf{V} = Ve^{i\alpha} = V_x + iV_y = \overline{w'(z)} \\ &V = |w'(z)|, \ \alpha = -\arg w'(z), \\ &\mathbf{V} = \operatorname{grad}\phi. \end{aligned}\right\} \tag{2}$$

Let C be a closed contour traversed in the positive direction (the contour C may also consist of the two sides of an arc traversed in opposite directions). The quantity

$$\Gamma = \int_C V_s \, ds = \int_C V_x \, dx + V_y \, dy = \int_C d\phi \tag{3}$$

is known as the *circulation* of the vector $\mathbf{V}$ with respect to the contour C.

The quantity

$$Q = \int_C V_n \, ds = \int_C (-V_y \, dx + V_x \, dy) = \int_C d\psi \tag{4}$$

(n is the outward normal to the closed contour C, traversed in the positive direction) is called the *flow* of the vector $\mathbf{V}$ through the contour C. Similarly the *flow* of the vector $\mathbf{V}$ through the arc AB is defined as the integral $\int_{AB} V_n \, ds$ (the direction of the normal n must be indicated).

Combining formulae (3) and (4), we obtain

$$\Gamma + iQ = \int_C w'(z) \, dz. \tag{5}$$

† For this chapter see [3, Chapter III] and the references given there.

If $w'(z)$ is defined within C and has a finite number of singularities there, then

$$\Gamma+iQ = 2\pi i \sum \operatorname{res} w'(z).$$

If a is a pole of the function $w'(z)$, then $w(z)$ possesses in the neighbourhood of a an expansion of the form

$$w(z) = \frac{c_{-n}}{(z-a)^n} + \ldots + \frac{p}{2\pi}\frac{1}{z-a} + \frac{\Gamma+iQ}{2\pi i}\log(z-a) + c_0 + c_1(z-a) + \ldots$$

It is said that the term $\frac{\Gamma+iQ}{2\pi i}\log(z-a)$ (Γ, Q, real numbers) defines a *vortex source* of strength Q and intensity Γ, denoted by $(a; Q, \Gamma)$†, the term $\frac{p}{2\pi}\frac{1}{z-a}$ is a *doublet* with moment p, denoted by $(a; p)$ (p is a complex number; the argument of p determines the direction of the axis of the doublet passing through the point a in the direction of a stream line), the remaining terms $(C_{-k})/(z-a)^k$ define at the point a *multiplets* of order $2k$.

Correspondingly, if at ∞

$$w(z) = c_n z^n + \ldots + \frac{p}{2\pi}z + \frac{\Gamma+iQ}{2\pi i}\log z + c_0 + \frac{c_{-1}}{z} + \ldots,$$

then the term $\frac{\Gamma+iQ}{2\pi i}\log z$ defines at ∞ a *vortex source* of strength Q and intensity Γ, the term $pz/2\pi$ defines a *doublet* with moment p (the direction of the stream line at infinity coincides with the argument of p), the remaining terms $c_k z^k$ define *multiplets* of order $2k$.

Points at which $\mathbf{V} = 0$, and consequently $w'(z) = 0$, are known as *stagnation points* of the flow; from these points stream lines and equipotential lines issue alternately. If a stagnation point is a zero of order $(n-1)$, these curves enclose an angle π/n. This branching of the curves may also occur at infinity.

In problems **1376–1389** it is required from the given complex potential of the flow to construct the equipotential lines and stream lines, to determine $\mathbf{V}$, the singularities and stagnation points, the strength and intensity of the vortex sources, the moments of the doublets and also to investigate the behaviour of the flow at ∞.

1376. $w = cz$ $(c = \alpha + i\beta)$.

1377. $w = z^n$ (in particular, $n = 2, 3$).

1378. $w = \frac{\Gamma+iQ}{2\pi i}\log z$. Consider, in particular, the cases $\Gamma = 0$ and $Q = 0$.

1379. $w = \frac{\Gamma+iQ}{2\pi i}\log\frac{z-a}{z-a}$.

† If $Q = 0$, then we have a *vortex* $(a; \Gamma)$. If $\Gamma = 0$, we have a *source* $(a; Q)$. If the intensity of the source $Q < 0$, it is often described as a *sink*.

1380. $w = 1/z$. Determine also the velocity at the points $2 \pm i$.

1381. (1) $w = z + \dfrac{R^2}{z}$; (2) $w = z - \dfrac{R^2}{z}$.

1382. $w = 1/z^2$.

1383. $w = \log(z^2 - a^2)$ $(a > 0)$. Determine also the velocity at the points $\pm ia$.

1384. $w = \log \dfrac{z^2 - a^2}{z^2 + a^2}$ $(a > 0)$.

1385. $w = \dfrac{Q}{2\pi} \log \left(z - \dfrac{1}{z} \right)$.

1386. $w = \log \left(1 + \dfrac{1}{z^2} \right)$.

1387. $w = \log \left(z^2 + \dfrac{1}{z^2} \right)$.

1388. $w = az + \dfrac{Q}{2\pi} \log z$ $(a > 0,\ Q > 0)$.

1389. $w = az + \dfrac{\Gamma}{2\pi i} \log z$ $(a > 0,\ \Gamma > 0)$.

1390. Investigate the character of the flow in the region $|z| \geqslant R$, if

$$w = a \left(z + \frac{R^2}{z} \right) + \frac{\Gamma}{2\pi i} \log z \quad (a > 0,\ \Gamma > 0).$$

Consider the cases: $\Gamma < 4\pi aR$, $\Gamma = 4\pi aR$, $\Gamma > 4\pi aR$.

1391. Find the complex potential $w(z)$ of the flow in the whole plane formed by the vortex sources $\{(a_k; Q_k, \Gamma_k)\}$ $(k = 1, 2, \ldots, n)$ and having at infinity the given velocity $\mathbf{V}_\infty = Ve^{i\alpha}$.

1392. Is it possible for stream lines to emerge from a point having: (1) a vortex, (2) a doublet, (3) a vortex and a doublet together?

1393. Find the law of variation of a vortex source, doublet and multiplet, situated at a point a or at ∞, for the following schlicht conformal mappings of the neighbourhood of these points $(c_1 \neq 0, c_{-1} \neq 0)$:

(1) $\zeta = \alpha + c_1(z - a) + \ldots$;

(2) $\zeta = \alpha + \dfrac{c_{-1}}{z} + \ldots;$

(3) $\zeta = \dfrac{c_{-1}}{z-a} + c_0 + \ldots;$

(4) $\zeta = c_1 z + c_0 + \ldots$

1394. Find the law of variation of the vortex source for the n-sheeted mappings:

$$\zeta = \alpha + c_n(z-a)^n + \ldots, \quad c_n \neq 0;$$

$$\zeta = \alpha + \frac{c_{-n}}{z^n} + \ldots, \qquad c_{-n} \neq 0.$$

In problem **1395** there is established the *symmetry principle*, which together with conformal mapping is widely used for the construction of flows (see problems **1397–1405**).

1395. Prove that the flow can be continued by the symmetry principle across a rectilinear or circular segment of a stream line or equipotential line, a vortex source passing into a vortex source, a doublet into a doublet, a multiplet, generally speaking, into a set of multiplets of the same order. Find the strength and intensity of the reflected vortex source and the moment of the reflected doublet.

REMARK. It follows from the symmetry principle that if there are rectilinear or circular segments on the stream line or equipotential line the flow must be symmetrical with respect to this curve. This imposes a definite restriction not only on the singularities of the flow outside the given lines but also on them or at their end points (if any).

1396. A flow in the z-plane is formed of a finite number of sources, vortices and doublets.

(1) Find the necessary and sufficient condition for the circle $|z| = R$ to be a stream line, if the sources, vortices, doublets:

(a) are not situated on this circle;

(b) are all situated on it;

(c) some arc on it and some not.

(2) With the same assumptions find the conditions for the circle $|z| = R$ to be an equipotential line.

1397. Find the complex potentials of the flows in the upper half-plane $\operatorname{Im} z > 0$ with the given singularities and the velocity $\mathbf{V}_\infty$.

(1) The velocity $\mathbf{V}_\infty = V$.

(2) The vortex $(a; \Gamma)$ and velocity $\mathbf{V}_\infty = 0$.

(3) The source $(a; Q)$ and velocity $\mathbf{V}_\infty = 0$.

(4) The doublet $(a; p)$ and velocity $\mathbf{V}_\infty = 0$.

(5) The vortex sources $\{(a_k; Q_k, \Gamma_k)\}$ $(k = 1, 2, \ldots, n)$, the doublet $(a; p)$ and velocity $\mathbf{V}_\infty = V$. What can be said of the behaviour of the flow at ∞?

(6) The vortex source $(0; Q, \Gamma)$ and doublet $(0; p)$; $\mathbf{V}_\infty = 0$. What values can the moment p of the doublet assume? Is the flow always possible if $\Gamma \neq 0$?

1398. In the circle $|z| < R$ construct flows, having respectively: (1) a vortex $(a; \Gamma)$; (2) a doublet $(a; p)$.

1399. Find conditions for the possibility of constructing flows in the circle $|z| < R$, if:

(1) There exist only the sources $\{(a_k; Q_k)\}$ $(k = 1, 2, \ldots, n,)$ situated inside the circle;

(2) In addition to the sources of part (1) there are sources $\{(a'_k; Q'_k)\}$ $(k = 1, 2, \ldots, m)$, situated on the circumference $|z| = R$.

In both cases find the complex potentials of the flows.

1400. In the domain $|z| > R$ construct flows having respectively:

(1) The vortex $(a; \Gamma)$, the velocity $\mathbf{V}_\infty = 0$ and cireulation at infinity $\Gamma_\infty = 0$;

(2) The doublet $(a; p)$, the velocity $\mathbf{V}_\infty = 0$ and circulation $\Gamma_\infty = 0$;

(3) The velocity $\mathbf{V}_\infty = Ve^{i\alpha}$ and circulation $\Gamma_\infty = 0$;

(4) The velocity $\mathbf{V}_\infty = Ve^{i\alpha}$ and eirculation Γ round the circle $|z| = R$.

REMARK. The last two examples of problem **1400** give the streamlining of a circle with given velocity at infinity, without circulation and with circulation (see, for example, [3, Chapter III, § 49]).

In examples **1401–1405**, using the symmetry principle construct flows with given singularities (at infinity and at angular points the veloeity is equal to zero).

1401. In the domain $|z| > 1$, $\operatorname{Im} z > 0$, with the vortex $(ia; \Gamma)$, $a > 0$.

1402. In the angle $0 < \arg z < \pi/3$, with the source $(ae^{i\frac{\pi}{6}}; Q), a > 0$.

1403. In the first quadrant $\operatorname{Re} z > 0$, $\operatorname{Im} z > 0$, with the source $(1; Q)$.

1404. In the first quadrant $\operatorname{Re} z > 0$, $\operatorname{Im} z > 0$, with the source $(1; Q)$ and the sink $(i; -Q)$.

1405. In the first quadrant $\operatorname{Re} z > 0$, $\operatorname{Im} z > 0$, with the source $(1+i; Q)$ and the sink $(0; -Q)$.

1406. Construct a flow in the whole z-plane, if it is known that in the upper half-plane $\operatorname{Im} z > 0$ there are vortex sources $\{(a_k; Q_k, \Gamma_k)\}$ $(k = 1, 2, \ldots, n)$ and a doublet $(a; p)$, the x-axis is an equipotential line and the velocity $\mathbf{V}_\infty = Ve^{i\alpha}$. Is such a flow always possible?

1407. Construct a flow in the whole z-plane if it is known that in the circle $|z| < R$ there are vortex sources $\{(a_k; Q_k, \Gamma_k)\}(k = 1, 2, \ldots, n)$ and a doublet $(a; p)$, the circle $|z| = R$ is an equipotential line and the velocity $\mathbf{V}_\infty = Ve^{i\alpha}$. Is such a flow always possible?

1408. In the simply connected domain D, bounded by the contour C, construct a flow with streamline C, having the vortex sources $\{(a_k; Q_k, \Gamma_k)\}$ $(k = 1, 2, \ldots, n)$. Is such a flow always possible?

1409. In the domain D, bounded by the contour C and containing the point at infinity, construct a flow with stream line C, which has the vortex sources $\{(a_k; Q_k, \Gamma_k)\}$ $(k = 1, 2, \ldots, n)$ and the given velocity $\mathbf{V}_\infty = Ve^{i\alpha}$. Is such a flow always possible?

In problems **1410–1417** the streamlining of bounded and unbounded contours (they must be stream lines) is considered. The problems are solved by means of conformal mapping onto the exterior of a circle, the upper half-plane and a rectilinear strip.

1410. Construct the streamlining of a bounded contour C with given circulation Γ and velocity $\mathbf{V}_\infty = Ve^{i\alpha}$. What is the mapping effected by the complex potential $w(z)$ in the case $\Gamma = 0$?

1411. Construct the streamlining of the ellipse $x^2/a^2 + y^2/b^2 = 1$:

(1) With given velocity $\mathbf{V}_\infty$, without circulation;

(2) With given velocity $\mathbf{V}_\infty$ and circulation Γ.

1412. Construct the streamlining of a flat plate $|x| \leqslant C$, $y = 0$:

(1) With given velocity $\mathbf{V}_\infty$, without circulation;

(2) With given velocity $\mathbf{V}_\infty$ and circulation Γ, defined by the condition that one of the ends of the plate is a point of departure of the flow (the Zhukovskii–Chaplygin postulate).

1413. Construct the streamlining of a Zhukovskii† profile with given velocity $\mathbf{V}_\infty$ and circulation Γ, defined by the Zhukovskii–

† See problem 274.

Chaplygin postulate (the sharp end of the profile must be a point of departure).

In problems **1414-1417** construct the streamlining of the given contours.

1414. The parabolas $y^2 = 2px$ (from the inside and from the outside).

1415. The right branch of the hyperbola $x^2/a^2 - y^2/b^2 = 1$ (from the inside, and from the outside with the velocity $V_\infty = 0$).

1416. The half lines: $-\infty < x < -1$, $y = \pm\pi$.

1417. The half lines: $1 < |x| < \infty$, $y = 0$.

In problems **1418–1422** periodic flows $\{V(z+\omega) = V(z)\}$ and flows in curvilinear strips (channels) are considered. For the construction of these flows the curvilinear strips must be conformally mapped onto a rectilinear strip, and then the flow continued by the symmetry principle and use made of the expansion of a meromorphic function into a series of partial fractions.

In problems **1418, 1419** investigate the singularities, construct schematically stream lines and equipotential lines and determine the velocity at infinity in the strip of periods for periodic flows with the given complex potentials.

1418. (1) $w = \dfrac{Q}{2\pi} \log \sin z$; (2) $w = \dfrac{\Gamma}{2\pi i} \log \sin z$.

1419. $w = \dfrac{p}{2\pi} \cot z \left(0 \leqslant \arg p \leqslant \dfrac{\pi}{2}\right)$.

1420. In the reetilinear strip $S: 0 < x < \omega$ of the z-plane construct the flow formed by the vortex source $(a; Q, \Gamma)$, $a \in S$, having the given velocities $V(x+i\infty) = V_1$, $V(x-i\infty) = V_2$. Is such a flow always possible?

Construct schematically stream lines and equipotential lines if $\Gamma = 0$ or $Q = 0$.

HINT. Continue the flow by the symmetry principle and use the result of problem **1418**.

1421. In a rectilinear strip of the z-plane $S: 0 < x < \omega$ construct the flow formed by the doublet $(a;p)$, $a \in S$, having the given velocity $V(x \pm i\infty) = iV$. Construct schematically the stream lines and equipotential lines.

1422. In a eurvilinear strip of the z-plane S, bounded by contours C_1, C_2, construct the flow streamlining C_1, C_2, having

given vortex sources, and doublets in S and given velocities $\mathbf{V}_1$, $\mathbf{V}_2$ at the points at infinity Ω_1, Ω_2 of the strip S. Indicate sufficient conditions for the existence of such a flow.

A flow is said to be *doubly periodic* if its velocity $\overline{w'(z)}$ is an elliptic function.

Elliptic functions are *doubly periodic* meromorphic functions, possessing periods 2ω and $2\omega'$, where $\operatorname{Im}(\omega'/\omega) \neq 0$ (in what follows it is assumed that $\operatorname{Im}(\omega'/\omega) > 0$). It follows from this definition that

$$f(z+2m\omega+2n\omega') = f(z),$$

where m and n are any integers or zero.

The parallelogram with vertices z_0, $z_0+2\omega$, $z_0+2\omega'$, $z_0+2\omega+2\omega'$ (z_0 is an arbitrary point) is known as the *parallelogram of periods*.

If $f(z)$ is an elliptic function different from a constant, then it possesses the following properties (Liouville's theorem):

(1) $f(z)$ has at least one pole in the parallelogram of periods;

(2) The sum of the residues of the function $f(z)$ at all the poles situated in the period parallelogram is equal to zero;

(3) In the period parallelogram the equation $f(z) = a$ has the same number of roots for any complex number a, finite or infinite (this number of roots is known as the *order* of the elliptic function);

(4) The difference between the sum of all the zeros and the sum of all the poles of the function $f(z)$, situated in the period parallelogram is equal to some period of it, that is,

$$\sum \alpha_k - \sum \beta_k = 2\mu\omega+2\nu\omega' \quad (\mu \text{ and } \nu \text{ are integers}).$$

The *Sigma function of Weierstrass* is defined as the integral function

$$\sigma(z) = z \prod{}' \left(1-\frac{z}{\Omega}\right) e^{\frac{z}{\Omega}+\frac{z^2}{2\Omega^2}}, \tag{6}$$

where $\Omega = 2n\omega+2m\omega'$ and the product extends to all Ω different from zero. The function $\sigma(z)$ is odd.

The *zeta function of Weierstrass* is the meromorphic function

$$\zeta(z) = \frac{\sigma'(z)}{\sigma(z)} = \frac{1}{z} + \sum{}' \left(\frac{1}{z-\Omega}+\frac{1}{\Omega}+\frac{z}{\Omega^2}\right), \tag{7}$$

where the summation extends to all Ω different from zero. The function $\zeta(z)$ is odd.

The Weierstrass function $\mathfrak{p}(z)$ with periods 2ω and $2\omega'$ (see page 236) is connected with $\zeta(z)$ by the relation $\mathfrak{p}(z) = -\zeta'(z)$.

Since

$$[\zeta(z+2\omega)-\zeta(z)]' = \mathfrak{p}(z)-\mathfrak{p}(z+2\omega) = 0,$$

it follows that

$$\zeta(z+2\omega)-\zeta(z) \equiv 2\eta$$

and similarly

$$\zeta(z+2\omega')-\zeta(z) \equiv 2\eta'$$

where η and η' are constants. Using the oddness of the function $\zeta(z)$, it is easy to prove that $\eta = \zeta(\omega)$ and $\eta' = \zeta(\omega')$.

The quantities η, η', ω and ω' are connected by Legendre's relation

$$\eta\omega'-\eta'\omega = \pi i/2.$$

Let us use the notation

$$\eta = \eta_1, \quad \eta' = \eta_3 \text{ and } \quad \eta+\eta' = \eta_2,$$

$$\omega = \omega_1, \quad \omega' = \omega_3 \text{ and } \quad \omega+\omega' = \omega_2.$$

The functions $\sigma_k(z)$ are determined by the relations

$$\sigma_k(z) = -e^{\eta_k z}\frac{\sigma(z-\omega_k)}{\sigma(\omega_k)} \qquad (k = 1, 2, 3). \tag{8}$$

Correspondingly

$$\zeta_k(z) = \frac{\sigma_k'(z)}{\sigma_k(z)}. \tag{9}$$

The functions $\sigma_k(z)$ are connected with Weierstrass's $\mathfrak{p}(z)$ and the functions sn z, cn z, dn z of Jacobi by the following formulae†.

$$\sqrt{[\mathfrak{p}(z)-e_k]} = \frac{\sigma_k(z)}{\sigma(z)}, \tag{10}$$

$$\operatorname{sn} u = \sqrt{(e_1-e_3)}\frac{\sigma(z)}{\sigma_3(z)}, \quad \operatorname{cn} u = \frac{\sigma_1(z)}{\sigma_3(z)}, \quad \operatorname{dn} u = \frac{\sigma_2(z)}{\sigma_3(z)}, \tag{11}$$

where $u = z\sqrt{(e_1-e_3)}$.

It is possible to express any elliptic function in terms of $\sigma(z)$ and $\zeta(z)$.

If $f(z)$ has only the simple poles b_k with residues A_k ($k = 1, 2, \ldots, n$), then

$$f(z) = \sum_{k=1}^{n} A_k\zeta(z-b_k)+C. \tag{12}$$

If $f(z)$ has the zeros a_k and poles b_k (of any multiplicity), then

$$f(z) = C\frac{\sigma(z-a_1)\sigma(z-a_2)\ldots\sigma(z-a_n)}{\sigma(z-b_1)\sigma(z-b_2)\ldots\sigma(z-b_n^*)}, \tag{13}$$

where

$$b_n^* = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n-1} b_k.$$

† $e_k = \mathfrak{p}(\omega_k)$. See page 236.

(2) $f(u+2\omega)=f(u)$; sketch the stream lines and equipotentials and investigate the mapping $t=f(u)$.

1424. Show that the flows defined by the complex potentials $\zeta_k(u)$ $(k=2,3,4)$, reduce to the flows of problem **1423** (for $C=0$) by means of displacements in the u and ζ-planes.

HINT. Make use of formulae (8) and (9).

1425. Show that the flows defined by the complex potentials

$$\theta_k'\left(\frac{u}{2\omega}\right)\Big/\theta_k\left(\frac{u}{2\omega}\right) \quad (k=2,3,4) \quad \text{and} \quad Z(u)$$

(see page 232), are reduced to the flows of problem **1423**, (2) by means of linear transformations.

1426. Show that the flow defined by the complex potential $E(u)$ (see page 232) is reduced to the flow of problem **1423**, (1) by means of linear transformations.

HINT. As a preliminary prove the relations

$$K^2=(e_1-e_3)\omega^2 \text{ and } \frac{E}{K}=\frac{e_1}{e_2-e_3}+\frac{\omega}{K^2}\eta.$$

1427. Find the complex potential $f(u)$ of the doubly periodic flow with two doublets $(\alpha;M)$, $(\beta;N)$ in the period parallelogram.

Explain the conditions for $f(u)$ to be an elliptic function and for the curves $\operatorname{Im} u=\pm\operatorname{Im}\omega'$, $\operatorname{Re} u=\pm\omega$ to be stream lines and equipotentials (or conversely); sketch the stream lines and equipotentials.

In problems **1428–1430** investigate the doubly periodic flows defined by the given complex potentials $f(u)$.

1428. $\operatorname{sn} u$.

1429. $\operatorname{cn} u$.

1430. $\operatorname{dn} u$.

1431. Find the complex potential $f(u)$ of the doubly periodic flow with two vortex sources $(\alpha;Q,\Gamma)$, $(\beta;-Q,-\Gamma)$ in the period parallelogram. In particular, consider the case $\alpha=0$, $\alpha=\omega$, $\alpha=\omega+\omega'$ and $\beta=\omega'$.

Find the form of the function $f(u)$ which satisfies the condition $f(u+2\omega)=f(u)$.

In problems **1432–1434** investigate the flows defined by the given complex potentials $f(u)$.

1432. (1) $\log \operatorname{sn} u$; (2) $\log \operatorname{cn} u$; (3) $\log \operatorname{dn} u$.

1433. $\mathfrak{p}(u)$.

1434. $\log \theta_k(v)$ $\left(v = \dfrac{u}{2\omega},\ k = 1, 2, 3, 4\right)$.

For the construction of a complex potential $f(z)$ in a doubly connected domain D it is usually conformally mapped onto a circular ring $R: \varrho < |t| < 1$ ($\mu = 1/\varrho$ is the modulus of D); in its turn the ring R with the radial cut $[\varrho, 1]$ is mapped by means of the function $t = e^{\frac{\pi u i}{\omega}}$ onto the rectangle with vertices 0, 2ω, $2\omega+\omega'$, ω' in the u-plane in such a way that the edges of the cut become the vertical sides and $\tau = \omega'/\omega = i/\pi \log 1/\varrho$. The characteristics of the flow in the rectangle are determined by the method of solution of problem **1393**. Since the bases of the rectangle are stream lines, the flow is continued through them by the symmetry principle (see problem **1395**), after which the complex potential $\Phi(u)$ of the resulting doubly periodic flow with periods 2ω, $2\omega'$ is determined; then $f(z) = \Phi[u(z)]$ (see the book by L. I. Sedov mentioned on page 242).

1435. Find the complex potential of the flow:

(1) In the circular ring $R: r_1 < |z| < r_2$ with circulations Γ along the boundary circles;

(2) In the arbitrary bounded doubly connected domain D† with circulation Γ along the boundary contours;

(3) In the exterior of two circles lying outside one another with circulations $\pm\Gamma$ on the boundary circles and with the condition $\mathbf{V}_\infty = 0$;

(4) In the doubly connected domain D, containing the point at infinity, with the circulations $\pm\Gamma$ along the boundary contours and with the condition $\mathbf{V}_\infty = 0$.

1436. Construct the flow in the circular ring $R: \varrho < |z| < 1$, formed by the doublet $(a;p)$ $(\varrho < a < 1)$ and streamlining without circulation the boundary contours. Investigate the mapping $t = f(z)$ and sketch the positions of the stream lines.

HINT. Make use of the solutions of problems **1393**,(1) and **1427**.

1437. Construct the flow in the doubly connected domain D, containing the point at infinity, which streamlines without circulation the boundary contours and has at infinity the given velocity $\mathbf{V}_\infty = Ve^{i\alpha}$.

† Here and in what follows it is assumed that functions mapping the domain D onto the ring are known.

1438. Construct the flow formed in the circular ring R: $\varrho < |z| < 1$, by a doublet and quadruplet located at the point $z = 1$ and streamlining the boundary circles without circulation. Sketch the distribution of stream lines and investigate the mapping $t = f(z)$. In particular, consider the case of a doublet only.

HINT. Write the complex potential $f(z)$ in the form

$$f(z) = \frac{c_{-2}}{(z-1)^2} + \frac{c_{-1}}{(z-1)} + c_0 + c_1(z-1) + \ldots$$

and explain what values of c_{-2} and c_{-1} are possible.

1439. (1) In the circular ring $R: \varrho < |z| < 1$ construct the flow formed by the vortex $(a; \Gamma)(\varrho < a < 1)$ and streamlining the boundary circles with circulations Γ_1 (along the circle $|z| = 1$) and Γ_2 (along the circle $|z| = \varrho$). Is it possible to specify Γ, Γ_1, Γ_2 arbitrarily? Consider, in particular, the cases $\Gamma_2 = 0$ and $\Gamma_2 = -\Gamma_1$.

Investigate the mapping effected by the functions $t = f(z)$, $\mathfrak{Z} = e^{\frac{2\pi i}{\Gamma} t}$ in the first case and by the functions $t = f(z)$, $\mathfrak{Z} = e^{\frac{-4\pi i}{\Gamma} t}$ and $s = \sqrt{(\mathfrak{Z} - \mathfrak{Z}_0)}$ $(\mathfrak{Z}_0 = -e^{\frac{4\pi\psi_0}{\Gamma}}$, where ψ_0 is the value of the stream function at the stagnation point) in the second case. Sketch the stream lines and equipotentials in the u-plane.

(2) In the doubly connected domain D, containing the point at infinity, construct the flow streamlining the boundary contours with given circulations Γ_1, Γ_2 and having at ∞ the velocity $\mathbf{V}_\infty = Ve^{i\alpha}$.

HINT. The solutions of problems **1437** and **1439**, (1) are superposed by means of conformal mapping.

Unsteady vortex flows†

Let $w(z)$ be a complex potential with the isolated singularities $\{a_k\}$ $(k = 1, 2, \ldots)$. If they are fixed we have a steady flow; if however, they are movable we have an unsteady flow in which the velocity at each singularity a_k is determined by the formula

$$\frac{da_k}{dt} = \left[\overline{\frac{dw_k(z)}{dz}}\right]_{z=a_k}$$

† For this section see: N.E. KOCHIN, I.A. KIBEL' and N.V. ROZE, *Theoretical hydromechanics* (*Teoreticheskaya gidromekhanika*), Chapter V, Gostekhizdat, 1948. For problems **1448–1450** see: F. OBERHETTINGER and W. MAGNUS, *Anwendungen der elliptischen Functionen in Physik und Technik*, Chapter IV, 1949.

where $w_k(z) = w(z) - S(z, a_k)$, $S(z, a_k)$ is the principal part of the expansion of $w(z)$ close to a_k, the logarithmic term also being included in this if a_k is a vortex source, that is, the velocity of the point a_k is determined by the action on it of all the remaining singularities. In this case $a_k = a_k(t)$ and the complex potential $w(z)$ depends on the time.

1440. Prove that two vortices $(z_1 = x_1+iy_1;\ \Gamma_1)$, $(z_2 = x_2+iy_2;\ \Gamma_2)$, acting on one another rotate round a "centre of inertia" $x_c = \dfrac{\Gamma_1 x_1+\Gamma_2 x_2}{\Gamma_1+\Gamma_2}$, $y_c = \dfrac{\Gamma_1 y_1+\Gamma_2 y_2}{\Gamma_1+\Gamma_2}$, if $\Gamma_1+\Gamma_2 \neq 0$ and move steadily perpendicular to the straight line connecting them if $\Gamma_1+\Gamma_2 = 0$.

1441. Along the real axis there is an impenetrable wall. At a distance h from it there is located a vortex of intensity Γ. Prove that it moves parallel to the x-axis with velocity $\dfrac{\Gamma}{4\pi h}$.

HINT. Apply the symmetry principle.

1442. Prove that a vortex located in a circle with an impenetrable boundary circumference, moves along a concentric circumference.

1443. Prove that n vortices of the same intensity Γ, arranged at a certain instant at the vertices of a regular n-gon inscribed in a given circle of radius R, move along this circle with constant speed $\dfrac{\Gamma(n-1)}{4\pi R}$.

1444. Prove that a vortex located in the first quadrant $x > 0, y > 0$ with impenetrable walls moves along the curve $\dfrac{1}{x^2} + \dfrac{1}{y^2} = C$.

1445. The flow of a fluid is caused by a vortex chain consisting of an infinite number of vortices of identical intensity Γ situated at a certain instant at the points $z_k = z_0+kl$ $(k = 0, \pm 1, \pm 2, \ldots)$. Prove the flow is steady with complex potential

$$w(z) = \frac{\Gamma}{2\pi i} \log \sin \frac{\pi}{l} (z-z_0).$$

REMARK. It is assumed that the velocity $V(x+iy)$ remains bounded as $y \to \pm \infty$. See problem **1420**.

1446. Prove that a vortex of intensity Γ, located in a rectilinear channel with impenetrable walls moves parallel to these walls with constant velocity $-\frac{\Gamma}{4l}\tan\frac{\pi a}{l}$ (l is the width of the channel, a is the distance from the vortex to the centre line of the channel). What is the flow if $a = 0$?

1447. The flow of a fluid is caused by two vortex chains consisting of vortices of intensity Γ_1, which at a certain instant are situated at the points $z_k^{(1)} = z_0^{(1)} + kl$, and vortices of intensity Γ_2 situated at the points $z_k^{(2)} = z_0^{(2)} + kl$ ($k = 0, \pm 1, \pm 2, \ldots$). Prove that the chains move with the same velocity when and only when $\Gamma_2 = -\Gamma_1$, this motion taking place parallel to the x-axis when and only when $z_0^{(2)} - z_0^{(1)} = ih$, the *symmetrical arrangement* or $z_0^{(2)} - z_0^{(1)} = \frac{l}{2} + ih$ the *chessboard arrangement* (h is an arbitrary real number); the speeds of motion in these cases are given respectively by $\frac{\Gamma}{2l}\operatorname{eoth}\frac{\pi h}{l}$ and $\frac{\Gamma}{2l}\tanh\frac{\pi h}{l}$ (Γ is the magnitude of the intensity of the vortex).

REMARK. It is assumed that as $y \to \pm\infty$ the velocity $\mathbf{V}(x+iy)$ remains bounded.

1448. The flow of a fluid in a rectilinear channel with impenetrable walls $x = \pm h$ is caused by a vortex chain consisting of vortices of intensity Γ, situated at a certain instant at the points $a + ikl$ ($k = 0, \pm 1, \pm 2, \ldots;\ -h < a < h$). Prove that the complex potential of the flow is given by

$$w(z) = \frac{\Gamma}{2\pi i}\log\frac{\theta_1\left(\frac{z-a}{4h}\right)}{\theta_1\left(\frac{z-2h+a}{4h}\right)},$$

where for the function θ_1 we have $2\omega = 4h$ and $2\omega' = il$, and that the vortex chain moves steadily parallel to the walls of the channel with constant speed $\frac{\Gamma}{2\pi}\left[\zeta(2h-2a) - \frac{\eta}{h}(h-a)\right]$. What is the flow if $a = 0$?

REMARK. At every instant the flow is doubly periodic with periods $2\omega = 4h$, $2\omega' = il$.

1449. The flow of a fluid in a rectilinear channel with impenetrable walls $X = \pm h$ is caused by two vortex chains consisting respectively of vortices of intensity Γ, and $-\Gamma$, situated at a certain instant at the points $a+ikl$, and $-a+i\left(\frac{l}{2}+kl\right)$ ($k = 0, \pm 1, \pm 2, \ldots$; $-h < a < h$), respectively, in chessboard order in the channel. Prove that the complex potential of the flow is given by

$$w(z) = \frac{\Gamma}{2\pi i}\left[\log\frac{\theta_1\left(\frac{z-a}{4h}\right)}{\theta_1\left(\frac{z-2h+a}{4h}\right)} - \log\frac{\theta_1\left(\frac{z+a-\frac{il}{2}}{4h}\right)}{\theta_1\left(\frac{z+2h-a-\frac{il}{2}}{4h}\right)}\right]$$

and that the vortex chains move steadily parallel to the walls of the channel with the constant velocity

$$-\frac{\Gamma}{2\pi}\left[2\zeta(2a) - 2\eta\frac{a}{h} + \frac{1}{2}\frac{\mathfrak{p}'(2a)}{\mathfrak{p}(2a)-e_1} + \frac{1}{2}\frac{\mathfrak{p}'(2a)}{\mathfrak{p}(2a)-e_3}\right].$$

What flow is obtained in the limit as $h \to \infty$?

HINT. In order to obtain the final expression for the speed of motion of the vortex chains use the addition formula of the ζ-function:

$$\zeta(u+v) = \zeta(u)+\zeta(v)+\frac{1}{2}\frac{\mathfrak{p}'(u)-\mathfrak{p}'(v)}{\mathfrak{p}(u)-\mathfrak{p}(v)}.$$

1450*. Find the motion of a vortex of intensity Γ, located in a rectangle with vertices 0, $2a$, $2bi$, $2a+2bi$ and with impenetrable walls. In particular, consider the case when at a certain instant the vortex is at the centre of the rectangle or close to it.

§ 2. Applications to electrostatics

A plane electrostatic field with intensity $\mathbf{E} = E_x+iE_y = Ee^{i\alpha}$ is characterised by the analytic function $w(z) = u+iv$, known as the *complex potential*; v is known as the *potential function* (it is always single valued), and u is the *force function*. The curves $v = \text{const}$ are the *equipotential lines*, and the

curves $u = \text{const}$ are the *lines of force* of the field. We have

$$\mathbf{E} = -\operatorname{grad} v = -\overline{iw'(z)}$$

$$E = |w'(z)|, \quad \alpha = -\pi/2 - \arg w'(z),$$

$$E_x = -\partial v/\partial x = \partial u/\partial y, \quad E_y = -\partial v/\partial y = -\partial u/\partial x.$$

In all the problems of this section where it is a question of electrostatic fields in domains bounded by one or several boundary contours it is assumed that the potential function is constant along each simple contour (that is, every such contour is a conductor).

If a is a pole of $w'(z)$ and close to a the function w has the expansion

$$w(z) = \frac{c_{-n}}{(z-a)^n} + \ldots + \frac{pi}{z-a} + 2qi \log \frac{1}{z-a} + c_0 + c_1(z-a) + \ldots,$$

then the term $2qi \log \dfrac{1}{z-a}$ defines at the point a a plane point *charge* of magnitude $\varrho = 2q$, denoted by $(a; 2q)$ (on unit length of a rectilinear conductor perpendicular to the z-plane at the point a, there occurs the charge q); the term $pi/(z-a)$ defines at the point a a *dipole* of moment p, denoted by $(a; p)$ (p is a complex number; its argument defines the direction of the axis of the dipole); the remaining terms $\dfrac{c_{-k}}{(z-a)^k}$ $(k = 2, 3, \ldots, n)$ define at the point a a *multipole* of order $2k$.

Correspondingly, if at ∞

$$w(z) = c_n z^n + \ldots + piz + 2qi \log z + c_0 + \frac{c_{-1}}{z} + \ldots,$$

then the term $2qi \log z$ defines at infinity a plane point charge of magnitude $\varrho = 2q$, and the term piz a dipole of moment p.

If the function $w = u + iv$ is considered as the complex potential of an electrostatic field $\mathbf{E} = -\overline{iw'(z)}$ and at the same time as the flow of a fluid with velocity $\mathbf{V} = \overline{w'(z)}$, then this leads to the following electrohydrodynamical analogy:

	Fluid flow	Electrostatic field
u	Potential function	Force function
$u = \text{const}$	Equipotential lines	Lines of force
v	Stream function (possibly many valued)	Potential function (always single valued)
$v = \text{const}$	Stream lines	Equipotential lines
$v_2 - v_1$	Outflow of fluid	Potential difference
$\oint du$	Circulation $\Gamma = \oint V_s \, ds$	Flux $N = \oint E_n \, ds$

	Fluid flow	Electrostatic field
—	Vortex $(a; \Gamma)$	Point charge $(a; 2q)$, $q = \frac{\Gamma}{4\pi}$
—	Source	—
—	Doublet with moment p	Dipole with moment $\frac{p}{2\pi i}$
	Streamlining with given vortices and doublets	Field with given charges, dipoles and equipotential boundary curves

In problems **1451–1458** it is required, from the given complex potentials, to determine the force and potential functions, the intensity of the field, the nature of the singularities (including that at ∞), and also to sketch the families of lines of force and equipotential lines (q is a real number). Compare with the solutions of problems **1376–1389**.

1451. $w = cz$ $(c = \alpha + i\beta)$.

1452. $w = 2qi \log \frac{1}{z}$.

1453. $w = 2qi \log \frac{z-a}{z-b}$.

1454. $w = 2qi \log (z^2 - a^2)$ $(a > 0)$.

1455. $w = pi/z$ $(p = |p|e^{i\alpha})$.

1456. $w = z \pm R^2/z$.

1457. $w = piz + 2qi \log \frac{1}{z}$ $(p > 0, q > 0)$.

1458. $w = piz + 2i \sum_{k=1}^{n} q_k \log \frac{1}{z - a_k}$ $(p > 0, \quad q_k > 0, \quad a_1 < a_2 < \ldots < a_n)$.

1459. Find the law of variation of the point charge $(a; 2q)$ and of the dipole $(a; p)$:

(1) In a single-valued conformal mapping;

(2) In continuation by the symmetry principle across a rectilinear or circular segment of an equipotential line.

1460. Show that the complex potential of the electrostatic field formed by the point charge $(a; 2q)$ in the arbitrary single-valued domain D is determined by the formula

$$w = 2qi \log \frac{1}{f(z,a)} + c,$$

where $f(z,a)$ is the function which conformally maps the domain D onto the unit circle in such a way that $f(a,a) = 0$, and c is a real constant.

Establish the connection between the potential function $v(z)$ and the Green's function of the domain D (see problem **1068**).

In problems **1461–1467**, using the results of problem **1460** or the symmetry principle, find the complex potentials of the electrostatic fields formed by the given point charges in the domains indicated.

1461. In the upper half-plane $\operatorname{Im} z > 0$, by the charge $(z_0; 2q)$.

1462. (1) In the circle $|z| < R$, by the charge $(z_0; 2q)$;

(2) In the exterior of the circle $|z| > R$, by the charge $(z_0; 2q)$.

1463. In the exterior of the ellipse $x^2/a^2 + y^2/b^2 = 1$, by the charge $(\infty; 2q)$.

1464. In the exterior of the segment $|x| < R$, $y = 0$, by the charge $(\infty; 2q)$.

1465. In the exterior of the square $|x| < d$, $|y| < d$ by the charge $(\infty; 2q)$.

1466. In the rectangle $|x| < a$, $|y| < b$, by the charge $(0; 2q)$.

1467. In the rectangle $0 < x < 2a$, $0 < y < 2b$, by the charge $(z_0; 2q)$.

In problems **1468–1472** construct the electrostatic fields formed by the given dipoles.

1468. In the circle $|z| < R$, by the dipole $(a; p)$.

1469. In the exterior of the circle $|z| > R$, by the dipole $(a; p)$.

1470. In the exterior of the segment $|x| < R$, $y = 0$, by the dipole $(\infty; p)$.

1471. In the exterior of the ellipse $x^2/a^2 + y^2/b^2 = 1$, by the dipole $(\infty; p)$.

1472. In the rectangle $|x| < a$, $|y| < b$, by the dipole $(0; p)$ $(pi = \varrho e^{i\alpha})$.

1173. Prove that the electrostatic field formed by the dipole $(a; p)$ in the arbitrary simply connected domain D, is deter-

mined by the complex potential $w = f(z)$, where the function $f(z)$ maps the domain D onto the exterior of a horizontal segment in such a way that $f(a) = \infty$, and the principal part of $f(z)$ at the point a equals $pi/(z-a)$ if $a \neq \infty$, and equals piz if $a = \infty$.

Find $f(z)$ if a known function $t(z)$ maps the domain D:

(1) Onto the interior of the unit circle, if $a \neq \infty$, where $t(a) = 0, t'(a) > 0$;

(2) Onto the exterior of the circle $|t| > R$, if $a = \infty$, where $t(\infty) = \infty$ and $t'(\infty) = 1$.

1474. Construct the electrostatic field formed by the point charges $\{(a_k; 2q_k)\}$ $(k = 1, 2, \ldots, n)$ and the dipoles $(a; p)$ in the simply connected domain D.

Let $g(\zeta, z)$ be the Green's function of the domain D (see page 165), the boundary Γ of which consists of the piecewise smooth simple contours $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$; also let n be the inward normal to Γ and let Γ be traversed in the positive direction with respect to D. If $u(z)$ is a function harmonic in the domain D and continuous on Γ, then it follows from Green's formula that

$$u(z) = \frac{1}{2\pi} \int_\Gamma u(\zeta) \frac{\partial g(\zeta, z)}{\partial n} ds$$

or

$$u(z) = \frac{1}{2\pi} \int_\Gamma \left[u(\zeta) \frac{\partial}{\partial n} \log \frac{1}{|\zeta - z|} - \log \frac{1}{|\zeta - z|} \frac{\partial u(\zeta)}{\partial n} \right] ds .$$

If the domain D contains the point at infinity and the function $u(z)$ is harmonic there, $u(\infty)$ must be added to the right hand sides of the given formulae. Then in the neighbourhood of the point at infinity the Green's function $g(z,\infty)$ can be represented in the form

$$g(z, \infty) = \log |z| + \gamma + o\left(\frac{1}{|z|}\right).$$

The quantity

$$\gamma = \lim_{z \to \infty} [g(z, \infty) - \log |z|]$$

is known as the Roben's constant of the closed set representing the complement of D on the z-plane; the quantity $e^{-\gamma}$ is known as the *capacity* of this set.

1475. Prove the following assertions (n is the inward normal):

(1) $g(z, a) = \log \frac{1}{|z-a|} - \frac{1}{2\pi} \int_F \frac{\partial g(\zeta, a)}{\partial n} \log \frac{1}{|\zeta - z|} ds$ if $a \neq \infty$;

(2) $g(z, \infty) = \gamma - \frac{1}{2\pi}\int_\Gamma \frac{\partial g(\zeta,\infty)}{\partial n} \log\frac{1}{|\zeta - z|}\,ds,$

if $z \in D$ and the domain D contains the point at infinity;

(3) $\frac{1}{2\pi}\int_\Gamma \frac{\partial g(\zeta,\infty)}{\partial n} \log\frac{1}{|\zeta - z|}\,ds = \gamma$ if $z \bar{\in} D$ and the domain D contains the point at infinity;

(4) $\frac{1}{2\pi}\int_\Gamma \frac{\partial g(\zeta, a)}{\partial n}\,ds = 1$, if $a \neq \infty$ or if $a = \infty \in D$.

HINT. In part (1) use the symmetry property of the Green function $g(\zeta, z) = g(z, \zeta)$ and the integral representation of the harmonic function in terms of its boundary values. In part (2) use the integral representation of the function $\log|\zeta - z| + g(\zeta, z) - g(\zeta, \infty)$, which is harmonic in D, pass to the limit and use the symmetry property $g(\infty, z) = g(z, \infty)$. In part (3) proceed in the same way but start with the function $\log|\zeta - z| - g(\zeta, \infty)$†.

The function

$$v_0(z) = 2q \log\frac{1}{|z - a|}$$

is known as the *logarithmic potential* of the point charge $(a; 2q)$. In the extended z-plane $v_0(z)$ represents the logarithmic potential of the two point charges: $(a, 2q)$ and $(\infty; -2q)$.

Let the contour Γ satisfy the conditions indicated on page 262, and let $\varrho(\zeta)$ and $\nu(\zeta)$ be real and continuous on Γ.

The integral

$$v(z) = \int_\Gamma \varrho(\zeta) \log\frac{1}{|\zeta - z|}\,ds$$

is known as the *logarithmic potential of the simple layer with density* $\varrho(\zeta)$ (in three dimensions there corresponds to it the potential of a charged cylindrical surface with base Γ and surface density of charge $\frac{\varrho}{2}$, that is bearing the charge $\frac{\varrho}{2}ds^2$ on the element of area ds^2).

The function $v(z)$ is continuous in the finite z-plane and is harmonic everywhere outside Γ, except at the point $z = \infty$, where it has a logarithmic singularity

$$v(z) = -2q \log|z| + o\left(\frac{1}{|z|}\right), \quad 2q = \int_\Gamma \varrho(\zeta)\,ds$$

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† See R. NEVANLINNA, *Eindeutige analytische Funktionen*, Chapter V, § 2, Springer, Berlin, 1936.

(this means that the potential $v(z)$ corresponds to a charge $(\infty; -2q)$).

The integral

$$v_1(z) = \int_\Gamma v(\zeta) \frac{\partial}{\partial n} \log \frac{1}{|\zeta - z|} \, ds$$

is known as the *logarithmic potential of the double layer with density* $v(\zeta)$ (if Γ is the boundary of a domain there is on Γ a distribution of dipoles with axes directed along the inward normal to Γ; $v(\zeta)$ is the density distribution of dipole moments). If $\theta(\zeta, z)$ is the angle between n and a vector going from ζ to z, and $d\phi(\zeta, z)$ is the angle subtended by the element of arc ds at z, then

$$v_1(z) = \int_\Gamma v(\zeta) \frac{\cos \theta(\zeta, z)}{|\zeta - z|} \, ds = \int_\Gamma v(\zeta) \, d\phi(\zeta, z).$$

In particular, for a closed contour Γ and $v(\zeta) \equiv 1$

$$\int_\Gamma \frac{\partial}{\partial n} \log \frac{1}{|\zeta - z|} \, ds = \begin{cases} 2\pi, & \text{if } z \text{ is inside } \Gamma, \\ \pi, & \text{if } z \text{ is on } \Gamma, \\ 0, & \text{if } z \text{ is outside } \Gamma \end{cases}$$

(see also problem **1066**).

The Green function $g(z, a)$ of the domain D can be considered as the potential of the electrostatic field formed by the point charge $(a; 1)$ if the boundary Γ of the domain D is earthed. Problem **1475**, (1) shows that in the case $a \neq \infty$ the earthing of Γ is equivalent to the placing on Γ of a charge of linear density $\varrho(\zeta) = -\frac{1}{2\pi} \frac{\partial g(\zeta, a)}{\partial n}$. In this case, by part (4) of problem **1475**, the total quantity of charge is equal to -1. In the case $a = \infty$ the point charge $(\infty; 1)$ and the earthing of Γ are together equivalent to the placing on Γ of a charge with density given by $\varrho(\zeta) = -\frac{1}{2\pi} \frac{\partial g(\zeta, \infty)}{\partial n}$ of total value -1 and to the addition of a field with constant potential γ (see **1475**, (2)). In these cases the given distributions on Γ are said to be *induced* by the charge $(a; 1)$.

In problems **1476–1479** find the density $\varrho(\zeta, a)$ of the distribution induced by the charge $(a; 1)$ on the contour Γ and the corresponding potential $v(z, a)$ of this simple layer for the domains bounded by the contour Γ.

1476. Γ is the real axis, $\operatorname{Im} a > 0$.

1477. (1) Γ is the circle $|z| = R$, $|a| < R$;

(2) Γ is the circle $|z| = R$, $|a| > R$ (consider, in particular, the case $a = \infty$).

1478. Γ is the real-axis segment $|x| \leqslant R$, $y = 0$, $a = \infty$.

1479. Γ is the ellipse $x^2/\alpha^2 + y^2/\beta^2 = 1$, $a = \infty$.

1480. Considering the Green function of the domain D containing the point $z = \infty$ to be known, solve *Roben's problem*:

Find the density of the distribution $\varrho(\zeta)$ on the boundary Γ of the domain D due to unit charge creating outside D and on Γ a constant potential†.

HINT. See problem **1475**, (3) and (4).

In problems **1481–1483** solve Roben's problem for the given domains D.

1481. D is the exterior of the circle $|z| > R$.

1482. D is the exterior of the segment $|x| \leqslant R$, $y = 0$.

1483. D is the exterior of the ellipse $x^2/a^2+y^2/b^2 = 1$.

In problems **1484–1487** find the capacity (see page 262) of the closed sets.

1484. $|z| \leqslant R$.

1485. $|x| \leqslant R$, $y = 0$.

1486. $x^2/a^2+y^2/b^2 \leqslant 1$.

1487. $|z^2-a^2| \leqslant a^2$ $(a > 0)$.

1488. On the simple closed contour Γ let there be given a real function $\phi(\zeta)$ continuous and differentiable along the contour.

Prove that the real part of the Cauchy type integral $\frac{1}{2\pi i}\int_\Gamma \frac{\phi(\zeta)}{\zeta-z}\mathrm{d}\zeta$ is the logarithmic potential of a double layer with density $\phi(\zeta)$ and its imaginary part is the logarithmic potential of a simple layer with density $-\frac{1}{2\pi}\frac{\mathrm{d}\phi}{\mathrm{d}s}$.

1489. Prove that the function $v(z)$, bounded and harmonic in the upper half-plane Im $z > 0$, can be represented as the logarithmic potential of a double layer:

$$v(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} v(t)\frac{\partial}{\partial n}\log\frac{1}{|t-z|}\mathrm{d}t.$$

If however $v(z)$ is regular at infinity, then it can also be represented as the logarithmic potential of a simple layer:

$$v(z) = v(\infty)-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\partial v(t)}{\partial n}\log\frac{1}{|t-z|}\mathrm{d}t.$$

† For the general Roben's problem whieh requires a non-negative distribution of unit charge on a given set E to be found in order that the corresponding logarithmie potential should assume one and the same value at every point of the set E, see the book by R. Nevanlinna mentioned on page 263.

1490. In the upper half-plane $\operatorname{Im} z > 0$ find the complex potential of an electrostatic field if its potential $v(z)$ assumes on the real axis given piecewise continuous values. Write the potential function in terms of the harmonic measures of the corresponding segments of the real axis (see page 170):

(1) ϕ in the interval $(-\infty, a)$, 0 in the interval (a, ∞);

(2) ϕ in the interval (a, b), 0 in the intervals $(-\infty, a)$, (b, ∞);

(3) $\phi_1, \phi_2, \ldots, \phi_n$ respectively in the intervals $(-\infty, a_1)$, (a_1, a_2), $\ldots$, (a_{n-1}, a_n) and 0 in the interval (a_n, ∞) (here $a_1 < a_2 < \ldots < a_n$);

(4) ϕ_0 in the interval (a_n, ∞), $\phi_1, \phi_2, \ldots, \phi_n$ respectively in the intervals $(-\infty, a_1)$, (a_1, a_2) and so on.

HINT. In part (1) use conformal mapping onto a strip; in the remaining parts use the method of superposition (it is possible also to use Schwarz's integral formula for the half-plane, see page 170).

1491. Find the complex potentials $w(z)$ and the potentials $v(z)$ in the following doubly connected domains with a given difference $d = v_2 - v_1$ of potentials v_1, v_2 on the boundary contours:

(1) In the circular ring $r_1 < |z| < r_2$;

(2) In an arbitrary doubly connected domain D.

1492. Prove that if D is an arbitrary doubly connected domain and on each of the contours bounding this domain the potential function assumes constant values (v_1 and v_2), then

$$w(z) = \frac{i(v_2 - v_1)}{\log \mu} \log t(z) + c + iv_1, \quad v(z) = \frac{v_2 - v_1}{\log \mu} \log |t(z)| + v_1,$$

where $t(z)$ conformally maps D onto the ring $1 < |t| < \mu$ (μ is the modulus of D) and the boundary contour with potential v_1 passes into the circle $|t| = 1$; c is a real number.

1493. Find the complex potentials in the given doubly connected domains (the potentials v_1 and v_2 on the boundary contours are constants).

(1) In the exterior of the circles $|z \pm a| = R$ $(a > R)$ (v_1 is the potential on the circumference of the circle on the left).

(2) In the exterior of the circles $|z| = r_1$ (potential v_1) and $|z - a| = r_2$ $(a > r_1 + r_2)$.

(3) In the non-concentric circular ring, bounded by the circles $|z| = R$ (the potential v_1) and $|z - a| = r$ $(0 < a < R - r)$.

(4) In the ellipse $x^2/a^2 + y^2/b^2 < 1$ with a cut along the segment joining the foci (the potential on the ellipse is v_1).

(5) In the exterior of the segments $1 < |x| < \frac{1}{k}$, $y = 0$ $(0 < k < 1)$. On the segment on the left the potential is v_1.

(6) In the exterior of the segments $|x| < 1$, $y = \pm\pi$. On the upper segment the potential is v_1.

1494. Let D be a multiply connected domain with boundary Γ, consisting of n piecewise smooth contours Γ_k $(k = 1, 2, \ldots, n)$, and let $\omega_k(z)$ be the harmonic measure of Γ_k (see page 170). If the domain D is bounded we shall consider Γ_n to be the exterior boundary contour. Prove the following assertions:

(1) If the domain D is bounded, then

$$\omega_k(z) = -\frac{1}{2\pi}\int_\Gamma \frac{\partial\omega_k(\zeta)}{\partial n}\log\frac{1}{|\zeta - z|}\,ds \quad (k = 1, 2, \ldots, n-1),$$

$$\omega_n(z) = 1 - \frac{1}{2\pi}\int_\Gamma \frac{\partial\omega_n(\zeta)}{\partial n}\log\frac{1}{|\zeta - z|}\,ds.$$

If the domain D contains the point at infinity, then

$$\omega_k(z) = \omega_k(\infty) - \frac{1}{2\pi}\int_\Gamma \frac{\partial\omega_k(\zeta)}{\partial n}\log\frac{1}{|\zeta - z|}\,ds \quad (k = 1, 2, \ldots, n).$$

(2) For points z not belonging to the domain D the right sides of the equations given in part (1) assume the value 1 in the domain complementary to D bounded by Γ_k (or Γ_n), and 0 in the domain complementary to D, bounded by the Γ_i $(i \neq k)$.

REMARK. By part (1) the functions $\omega_k(z)$ represent in the domain D potentials identical with the distributions induced by the charges distributed on Γ, with densities of the layer $\varrho_k(\zeta) = -\frac{1}{2\pi}\frac{\partial\omega_k(\zeta)}{\partial n}$. In the case of a bounded domain D the quantities $\omega_1(z), \ldots, \omega_{n-1}(z)$ exactly coincide with the logarithmic potentials of the given induced layers on Γ.

The values of the eharges of the layer p_{ik}, induced by the potential $\omega_k(z)$ on the contour Γ_i $(i, k = 1, 2, \ldots, n)$, that is,

$$p_{ik} = -\frac{1}{2\pi}\int_{\Gamma_i}\frac{\partial\omega_k(\zeta)}{\partial n}\,ds = -\frac{1}{2\pi}\int_\Gamma \omega_i\frac{\partial\omega_k(\zeta)}{\partial n}\,ds,$$

are known as the *mutual capacities* of the boundary eontours (some properties of the numbers p_{ik} are considered in problems **1101–1104**).

1495. Find the harmonic measures $\omega_k(z)$, and also the quantities $\varrho_k(\zeta)$ and p_{ik}, defined in the remark to the preceding problem, for:

(1) The circular ring $1 < |z| < \mu$;

(2) An arbitrary doubly connected domain D, assuming that the function which maps this domain onto the ring is known.

1496. Let D be the domain of problem **1494** and $v(z)$ the bounded potential of an electrostatic field which assumes the constant values α_k ($k = 1, 2, \ldots, n$) on the boundary contours (conductors) Γ_k. Prove the following assertions:

(1) $v(z) = \sum_{k=1}^{n} \alpha_k \omega_k(z)$.

(2) If the domain D is bounded and Γ_n is the external contour, then

$$v(z) = \alpha_n - \frac{1}{2\pi} \int_{\Gamma} \frac{\partial v(\zeta)}{\partial n} \log \frac{1}{|\zeta - z|} \, ds;$$

if, however, D contains the point ∞, then

$$v(z) = v(\infty) - \frac{1}{2\pi} \int_{\Gamma} \frac{\partial v(\zeta)}{\partial n} \log \frac{1}{|\zeta - z|} \, ds.$$

Prove that the right sides of these formulae are equal to α_k ($k = 1, 2, \ldots, n$) in the domains complementary to D bounded by the Γ_k.

(3) The values of the charges of the layer induced on the Γ_i are given by

$$2q_i = -\frac{1}{2\pi} \int_{\Gamma_i} \frac{\partial v(\zeta)}{\partial n} \, ds = \sum_{k=1}^{n} p_{ik} \alpha_k,$$

where

$$\sum_{i=1}^{n} q_i = 0.$$

HINT. See problem **1101**, (1).

(4) $$\frac{1}{2\pi} \iint_{D} (\operatorname{grad} v)^2 \, dx \, dy = \sum_{i,k=1}^{n} p_{ik} \alpha_i \alpha_k.$$

HINT. See problem **1102**.

(5) If $w(z)$ is the complex potential of the field the density of the induced layer is

$$\varrho(\zeta) = -\frac{1}{2\pi}\frac{\partial v(\zeta)}{\partial n} = \pm\frac{1}{2\pi}|w'(\zeta)|\,.$$

Let D be an arbitrary multiply connected domain with boundary Γ, consisting of the Jordan contours $\Gamma_1, \ldots, \Gamma_n$. There exist conformal mappings of the domain D onto each of the following canonical domains with the given uniqueness conditions (a, b are arbitrary points of the domain D, A is an arbitrary complex number):

(1) Onto the *plane with parallel cuts*. The mapping function $f(z)$ is uniquely determined by giving its pole a and the coefficient A of the expansion

$$f(z) = \begin{cases} \dfrac{A}{z-a} + c_1(z-a) + \ldots & (a \neq \infty)\,, \\ Az + \dfrac{c_{-1}}{z} + \ldots & (a = \infty)\,. \end{cases}$$

(2) Onto the *plane with radial cuts* (this refers to the plane with cuts along segments on rays originating at the coordinate origin, or with cuts along concentric circular arcs with centre at the coordinate origin). The function $f(z)$ is determined by its zero a, the pole b and the coeffieient A of the expansion

$$f(z) = \begin{cases} \dfrac{A}{z-b} + c_1(z-b) + \ldots & (b \neq \infty)\,, \\ Az + \dfrac{c_{-1}}{z} + \ldots & (b = \infty)\,. \end{cases}$$

(3) Onto a *disk with radial cuts* or *with cuts along concentric circular arcs* (centre at the coordinate origin). The function $f(z)$ is determined by the conditions

$$f(a) = 0, \quad f'(a) = 1$$

and the specification of a contour Γ_k, which is transformed into a circle.

(4) Onto the *annulus with radial cuts* or *with cuts along concentric circular arcs* (centre at the coordinate origin). The mapping is determined apart from a magnifieation and a rotation by the specification of the contours which are transformed into the interior and exterior boundary circles.

See, for example, G. M. Goluzin, *The geometrical theory of functions of a complex variable* (*Geometricheskaya teoriya funktsii kompleksnogo peremennogo*), Chapter V, Gostekhizdat, 1952.

1497. Assuming that the function which conformally maps the domain D onto the plane with parallel cuts, and the harmonic measures $\omega_k(z)$ of the boundary contours Γ_k $(k = 1, 2, \ldots, n)$†, are known, find the potential of the electrostatic field produced in the domain D by the dipole $(a; p)$ when the boundary Γ of the domain is earthed.

1498. Assuming that the Green function of the domain D is known, find the potential of the electrostatic field in this domain formed by the point charge $(a; 2q)$ $(a \in D)$ which has given potentials α_k on the boundary contours Γ_k $(k = 1, 2, \ldots, n)$. In the case when the domain D is simply connected obtain the formula of problem **1460.**

1499. Determine the nature of the electrostatic field determined in the multiply connected domain D by the complex potential $w = f(z)$, where $f(z)$ is the function which maps this domain onto the plane with cuts parallel to the real axis.

1500. Determine the nature of the electrostatic field defined in the domain D by the complex potential $w = 2qi \log \dfrac{1}{f(z)}$ if the function $f(z)$ maps the domain D onto:

(1) The plane with cuts along concentric circular arcs with centre at the coordinate origin;

(2) A disk with cuts along concentric circular arcs with centre at the coordinate origin;

(3) A circular annulus with cuts along concentric circular arcs with centre at the coordinate origin.

In each case find the flux (of the field intensity) through the boundary contours.

1501. Sketch the distribution of the equipotential lines and the lines of force of the electrostatic fields:

(1) Formed in an infinite doubly connected domain by a dipole at ∞;

(2) Formed in a bounded doubly connected domain D by a point charge.

In each case the potentials on the boundary contours are constants.

1502. (1) Express the potential $v(z)$ of the electrostatic field formed in the multiply connected domain D by the charge distribu-

† These are determined by means of Green's function; see, for example, §1 of the appendix by M. Schiffer to the book: R. COURANT, *Dirichlet's Principle, Conformal Mappings and Minimal Surfaces*, Interscience, New York, 1950.

tion $2q_k$ on $\Gamma_k\left(\sum_{k=1}^{n} q_k = 0\right)$ in terms of the harmonic measures $\omega_k(z)$ of its boundary contours. (The potential is constant on each contour Γ_k.)

HINT. Use the results of problem **1496**.

(2) Find the expression for the potential $v(z)$, if $\sum_{k=1}^{n} q_k = q \neq 0$ and there is a point charge $(a; -2q)$.

HINT. The determination of the potential $v(z)+2q\,g(z, a)$ reduces to part (1).

1503. Find the potential $v(z)$ in the circular ring $r_1 < |z| < r_2$, if the charge distributions on its contours are given as $2q_1$, and $2q_2$, where in the case $q_1+q_2 = q \neq 0$ there is also the point charge $(a; -2q)$.

HINT. The Green function of the circular ring can be determined from the solution of problem **1439** by choosing an appropriate circulation. From the Green functions one can determine the induced charges on the boundary contours.

§ 3. Applications to the plane problem of heat conduction

The plane problem of the steady distribution of temperatures in a field is characterised by the analytie function $w(z) = u+iv$ (u is the temperature), known as the *complex potential of a thermal field*. The vector $Q = -k \operatorname{grad} u = -k\overline{w'(z)}$ (k is the coefficient of thermal conductivity, constant in what follows) is known as the *vector heat flow*. The heat flow through the contour C is equal to

$$\int_C Q_n \mathrm{d}s = -k \int_C \frac{\partial u}{\partial n} \mathrm{d}s = -k \int_C \mathrm{d}v$$

(n is the outward normal to the contour C, traversed in the positive direction). Sinee the function u is single valued, for a closed contour C the heat flow is also given by $ik\int_C w'(z)\,\mathrm{d}z$. If close to the point a we have

$$w(z) = \left[\ldots + \frac{c_{-1}}{z-a} + c_0 + c_1(z-a) + \ldots\right] + \frac{q}{2\pi k} \log \frac{1}{z-a},$$

the term $\frac{q}{2\pi k} \log \frac{1}{z-a}$ defines at the point a a *source* $(a; q)$ of strength q, and the term $(c_{-1})/(z-a)$ defines a *doublet* at the point a.

The following similarities exist with fluid flow and the electrostatic fiield:

	Thermal field	Fluid flow	Electrostatic field
Complex potential	$w(z) = u+iv$	$w(z) = u+iv$	$iw(z) = -v+iu$
Vector field	$\mathbf{Q} = -k \operatorname{grad} u = -k\overline{w'(z)}$	$\mathbf{V} = \operatorname{grad} u = \overline{w'(z)}$	$\mathbf{E} = -\operatorname{grad} u = \overline{w'(z)}$
u	Temperature	Potential function	Potential function
$u = \text{const}$	Isothermals	Equipotential lines	Equipotential lines
v	Stream function	Stream function	$-v$ is the force function
$v = \text{const}$	Stream lines	Stream lines	Lines of force
	Source $(a; q)$	Source $\left(a; -\frac{q}{k}\right)$	Point charge $\left(a; \frac{q}{2\pi k}\right)$
	Doublet	Doublet	Dipole
	A thermal field with given sources, doublets and isothermal boundary contours	A flow determined by the complex potential $iw(z)$, with given vortices and doublets, streamlining the boundary contours	A field with given charges, dipoles and equipotential boundary contours.

1504. Formulate the symmetry principle for the continuation of a heat source across a rectilinear or circular segment of the boundary of a domain. Find the temperature distribution in an arbitrary simply connected domain D, if it is known that within this domain there is a source $(a; q)$ and the temperature on the boundary has the constant value C.

In problems **1505–1508** find the temperature distribution in the given domains from the given sources assuming that the temperature is constant on the boundary of the domain.

1505. In the upper half-plane $\operatorname{Im} z > 0$; the source $(a; q)$.

1506. In the circle $|z| < R$; the source $(a; q)$.

1507. In the half-strip $|x| < a$, $y > 0$ with the source $(ih; q)$ $(h > 0)$.

1508. In the rectangle $|x| < a$, $|y| < b$ with the source $(0; q)$.

1509. (1) Give an interpretation of the Green function $g(z, a)$ of the domain D in terms of the theory of heat conduction.

(2) Assuming the Green function of the domain D to be known find the temperature distribution in this domain if it is known that in D there is a source $(a; q)$ and on the boundary contours Γ_k $(k = 1, 2, \ldots, n)$ the temperature has the constant values u_k. Express the answer in terms of the harmonic measures $\omega_k(z)$ of the boundary contours.

1510. Find the temperature distribution inside the circular ring $r_1 < |z| < r_2$, if it is known that inside the ring there is the source $(a; q)$ and on the boundary circles the temperature has the constant values: u_1 on the circle $|z| = r_1$ and u_2 on the circle $|z| = r_2$.

HINT. See the similar problems **1439** and **1503**.

ANSWERS AND SOLUTIONS

CHAPTER I

1. (1) $-i$; (2) $-i$; (3) $\frac{1}{5}(1+3i)$; (4) -8.

2. (1) $3, \pi/2$ (here and below only the values of $\arg z$ are given);
(2) $2, \pi$; (3) $\sqrt{2}, \pi/4$; (4) $\sqrt{2}, -3\pi/4$; (5) $\sqrt{29}, \tan^{-1} 5/2$;
(6) $\sqrt{29}, -\tan^{-1} 5/2$; (7) $\sqrt{29}, \pi-\tan^{-1} 5/2$;

(8) $\sqrt{29}, \tan^{-1} 5/2-\pi$; (9) $|b|, \frac{\pi}{2}\frac{|b|}{b} = \frac{\pi}{2}\operatorname{sgn} b$†;

(10) $\sqrt{(a^2+b^2)}, \tan^{-1}\frac{b}{a}$ for $a > 0, \tan^{-1}\frac{b}{a} + \pi$ for $a < 0$ and $b \geqslant 0$,

$\tan^{-1}\frac{b}{a} - \pi$ for $a < 0$ and $b < 0$.

3. (1) $1, -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}$; (2) $\pm\frac{\sqrt{3}}{2} + \frac{i}{2}, \quad -i$;

(3) $\pm\frac{\sqrt{2}}{2}(1+i), \pm\frac{\sqrt{2}}{2}(1-i)$; (4) $\pm\frac{\sqrt{2}}{2}(\sqrt{(3)}+i), \pm\frac{\sqrt{2}}{2}(\sqrt{(3)}-i), \pm\sqrt{2}i$;

(5) $\pm 1 \pm i, \pm\frac{\sqrt{2}}{2}(1+i), \pm\frac{\sqrt{2}}{2}(1-i)$;

(6) $\pm\frac{\sqrt{2}}{2}(\sqrt{[\sqrt{(2)}+1]} - i\sqrt{[\sqrt{(2)}-1)}$; (7) $\pm(2+i)$;

(8) $\sqrt{2}\left[\cos\frac{\left(2k+\frac{3}{4}\right)\pi}{3} + i\sin\frac{\left(2k+\frac{3}{4}\right)\pi}{3}\right] \quad (k = 0, 1, 2)$;

(9) $\sqrt[5]{5}\left[\cos\frac{(2k+1)\pi-\tan^{-1}3/4}{5} + i\sin\frac{(2k+1)\pi-\tan^{-1}3/4}{5}\right]$
$(k = 0, 1, 2, 3, 4)$.

11. $z_k = z_1\left(\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}\right) \quad (k = 0, 1, 2, \ldots, n-1)$.

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† $\operatorname{sgn} b$ denotes Kronecker's symbol:

$$\operatorname{sgn} b = 1 \quad \text{for } b > 0,$$
$$\operatorname{sgn} b = -1 \quad \text{for } b < 0.$$

12. $z_3 = z_2 + (z_2 - z_1)\left(\cos\frac{2\pi}{n} \pm i \sin\frac{2\pi}{n}\right)$.

13. $z_4 = z_1 + z_3 - z_2$.

14. The ratio $(z_3 - z_1)/(z_2 - z_1)$ must be a real number (the condition is necessary and sufficient).

15. The anharmonic ratio $(z_1, z_2, z_3, z_4) = \frac{z_4 - z_3}{z_1 - z_4} : \frac{z_2 - z_3}{z_2 - z_4}$ must be a real number (the condition is necessary and sufficient).

16. SOLUTION. In the proof it can be assumed (without loss of generality) that the straight line in question is the imaginary axis and that all the points considered are on the right of it (if this is not the case it is necessary to multiply all the z_k by some number of the form $\cos\alpha + i\sin\alpha$). It is then obvious that $\operatorname{Re} z_k > 0$ and $\operatorname{Re}\frac{1}{z_k} > 0$ for any k.

19. The interior of the circle of radius R with centre at the point $z = z_0$; the exterior of the same circle; the circumference of the same circle.

20. An ellipse with foci at the points $z = \pm 2$ and major semi-axis 5/2.

21. The interior of the left hand branch of the hyperbola with foci at the points $z = \pm 2$ and real semi-axis 3/2.

22. The straight line perpendicular to the segment connecting the points z_1 and z_2, and passing through the middle of this segment.

23. (1) The straight line $x = C$ and the half-plane on the right of it; (2) the half-plane below the straight line $y = C$.

24. The strip $-1 < y < 0$.

25. The interior of the angle with vertex at the coordinate origin and sides which form with the real axis angles equal respectively to α and β; the interior of the same angle with vertex at the point z_0.

26. The parabola $y^2 = 2x + 1$.

27. The half-plane bounded by the straight line $x + y = 1$ and containing the coordinate origin.

28. The straight line passing through the points z_1 and z_2 (the point z_2 being omitted); the circle with the segment connecting the points z_1 and z_2 as diameter (the point z_2 being omitted).

29. (1) The interior of the domain bounded by the segment $0 \leqslant x \leqslant 2\pi$ of the real axis and one turn of the Archimidean spiral $r = \phi$; (2) The set of points determined in part (1) and the eomplement to the interval $(0, 2\pi)$ of the real axis.

30. (1) A family of circles touching the imaginary axis at the coordinate origin, and the imaginary axis itself (the equation of the family is: $C(x^2 + y^2) = x$); (2) The family of eircles touching the real axis at the coordinate origin and the real axis itself.

31. (1) The family of hyperbolas $x^2 - y^2 = C$; (2) The family of hyperbolas $xy = C/2$.

32. Each curve is a circle, being the geometrical locus of points the ratio of distances from which to the points z_1 and z_2 is constant (the eircle of Apollonius with respect to the points z_1 and z_2).

33. The family of circular arcs with ends at the points z_1 and z_2 (this family also contains the two rectilinear segments with ends at the points z_1 and z_2; one of these segments contains the point at infinity).

34. (1) Each curve is the geometrical locus of points, the product of the distances of which from the points $z = -1$ and $z = 1$ is constant (lemniscates with foci $z = \pm 1$). For $\lambda > 1$ the curves of the family are simple closed curves, for $\lambda < 1$ they separate into two simple closed curves which for $\lambda \to 0$ shrink to the points ± 1. For $\lambda = 1$ we have the lemniscate of Bernoulli; its equation in polar coordinates is $r^2 = 2\cos 2\phi$.

(2) Lemniscates with foci at the points z_1 and z_2, where z_1, z_2 are the of the equation $z^2+az+b = 0$. The lemniscates consist of a single curve if $\lambda > \sqrt{\left(\frac{|z_2-z_1|}{2}\right)}$, and of two curves if $\lambda < \sqrt{\left(\frac{|z_2-z_1|}{2}\right)}$. For $\lambda = \sqrt{\left(\frac{|z_2-z_1|}{2}\right)}$ we have a lemniscate of Bernoulli with the double point $(z_1+z_2)/2$.

35. The spiral of Archimides $r = \phi$.

36. The logarithmic spiral $r = e^{\phi}$.

37. (1) π; (2) $2\pi/3$; (3) 2π; (4) π; (5) 0.

38. $\xi = \frac{x}{x^2+y^2+1}$, $\eta = \frac{y}{x^2+y^2+1}$, $\zeta = \frac{x^2+y^2}{x^2+y^2+1}$; $z = x+iy = \frac{\xi+i\eta}{1-\zeta}$.

39. $\left(\frac{1}{2}, 0, \frac{1}{2}\right)$, $\left(-\frac{1}{2}, 0, \frac{1}{2}\right)$, $\left(0, \frac{1}{2}, \frac{1}{2}\right)$, $\left(\frac{\sqrt{2}}{4}, -\frac{\sqrt{2}}{4}, \frac{1}{2}\right)$.
All the four points are situated on the equator, their longitudes being respectively equal to 0, π, $\frac{\pi}{2}$, $-\frac{\pi}{4}$ (the longitudes are calculated from the initial meridian, situated in the ξ, ζ-plane).

40. The circle of radius $\tan(\beta/2+\pi/4)$ with centre at the point $z = 0$. The "South" pole corresponds to the coordinate origin, the "North" pole to the point at infinity.

41. (1) Half meridians of longitude α; (2) Parallels of latitude $\beta = 2\tan^{-1} r - \pi/2$.

42. (1) Diametrically opposite points on the same parallel; (2) Points mutually symmetrical with respect to the initial meridian (that is, differing in the sign of the longitudes); (3) Points mutually symmetrical with respect to the equatorial plane (that is, with the same longitude and with latitudes differing in sign only).

43. $z_1 \cdot \bar{z}_2 = -1$.

44. Points obtainable one from the other by a rotation of the sphere through 180° about a diameter parallel to the real axis of the z-plane.

45. (1) The eastern hemisphere; (2) The western hemisphere; (3) The hemisphere $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$ (α is the longitude); (4) The hemisphere $\frac{\pi}{2} < |\alpha| < \pi$; (5) The southern hemisphere; (6) The northern hemisphere.

46. A family of circles touching at the "north" pole (the pole of projections); a straight line passing through the coordinate origin corresponds to a great circle, and a straight line parallel to it at a distance d from the coordinate origin corresponds to a circle situated in a plane inclined at an angle $\tan^{-1} d$ to the meridional plane.

47. The straight line corresponds to a circle passing through the "North" pole.

50. $k(z, a) = \dfrac{|z-a|}{\sqrt{(1+|z|^2)}\sqrt{(1+|a|^2)}}$; $\quad k(z, \infty) = 1/\sqrt{(1+|z|^2)}$.

51. The circle of Apollonius $\left|\dfrac{z-z_1}{z-z_2}\right| = \dfrac{\sqrt{(1+|z_1|^2)}}{\sqrt{(1+|z_2|^2)}}$, in particular the straight line $|z-z_1| = |z-z_2|$, if $|z_1| = |z_2|$, and the circle $|z-z_1| = \sqrt{(1+|z_1|^2)}$, if $z_2 = \infty$.

54. $(a_1^2+b_1^2+c_1^2+d_1^2)(a_2^2+b_2^2+c_2^2+d_2^2) = (a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2)^2 + (a_1b_2+a_2b_1+c_1d_2-c_2d_1)^2 + (a_1c_2+a_2c_1+d_1b_2-d_2b_1)^2 + (a_1d_2+a_2d_1+b_1c_2 - b_2c_1)^2$.

55. SOLUTION. Let us multiply the arbitrary quaternion q and the quaternions qi, qj, qk by another arbitrary quaternion q_1. The components of the four quaternions obtained satisfy all the conditions of the problem except, perhaps, the diagonal condition. This last condition is satisfied by the choice of the components of the quaternions q and q_1. Here is an example of a matrix constructed by Euler:

$$\begin{pmatrix} 68 & -29 & 41 & -37 \\ -17 & 31 & 79 & 32 \\ 59 & 28 & -23 & 61 \\ -11 & -77 & 8 & 49 \end{pmatrix}$$

with the sum of the four squares equal to 8515.

57. 1, $e^{\pi i}$, $e^{\frac{\pi}{2}i}$, $e^{-\frac{\pi}{2}i}$, $\sqrt{2}\, e^{\frac{\pi}{4}i}$, $\sqrt{2}\, e^{-\frac{\pi}{4}i}$, $\sqrt{2}\, e^{\frac{3\pi}{4}i}$, $\sqrt{2}\, e^{-\frac{3\pi}{4}i}$.

58. $\pm i$; $(-1)^k$

59. e^2, 1; e^2, -3; e^3, $4-2\pi$; e^{-3}, $2\pi-4$; a, $\phi-\pi$, if $\phi > 0$, and $\phi+\pi$, f $\phi \leqslant 0$; 1, $-\phi$, if $|\phi| < \pi$, and π, if $\phi = \pi$.

60. (1) $\dfrac{\sin\frac{(n+1)x}{2}\cos\frac{nx}{2}}{\sin x/2}$; (2) $\dfrac{\sin\frac{(n+1)x}{2}\sin\frac{nx}{2}}{\sin x/2}$;

(3) $\dfrac{\sin 2nx}{2\sin x}$; (4) $\dfrac{\sin^2 nx}{\sin x}$; (5) $\dfrac{\sin\frac{(n+1)x}{2}\cos\frac{nx}{2}}{\cos x/2}$, if n is an odd number; $-\dfrac{\cos\frac{(n+1)x}{2}\sin\frac{nx}{2}}{\cos x/2}$, if n is an even number.

61. (1) $\dfrac{\sin\frac{n+1}{2}\beta}{\sin\beta/2}\cos\left(\alpha+\dfrac{n\beta}{2}\right)$; (2) $\dfrac{\sin\frac{n+1}{2}\beta}{\sin\beta/2}\sin\left(\alpha+\dfrac{n\beta}{2}\right)$.

65. (1) $\sin z = \sin(x+iy) = \sin x\cosh y + i\cos x\sinh y$, $|\sin z| = \sqrt{(\sinh^2 y+\sin^2 x)}$;

(2) $\cos z = \cos x\cosh y - i\sin x\sinh y$, $\quad |\cos z| = \sqrt{(\cosh^2 y-\sin^2 x)}$;

(3) $\tan z = \dfrac{1}{2}\dfrac{\sin 2x+i\sinh 2y}{\cos^2 x+\sinh^2 y}$, $\quad |\tan z| = \dfrac{\sqrt{(\sin^2 2x+\sinh^2 2y)}}{2(\cos^2 x+\sinh^2 y)}$;

(4) $\sinh z = \sinh x\cos y + i\cosh x\sin y$, $\quad |\sinh z| = \sqrt{(\sinh^2 x+\sin^2 y)}$;

(5) $\cosh z = \cosh x \cos y + i \sinh x \sin y$, $|\cosh z| = \sqrt{(\cosh^2 x - \sin^2 y)}$;

(6) $\tanh z = \dfrac{\sinh 2x + i \sin 2y}{\cosh 2x + \cos 2y}$, $|\tanh z| = \dfrac{\sqrt{(\sinh^2 2x + \sin^2 2y)}}{2(\sinh^2 x + \cos^2 y)}$.

66. (1) $\cos 2 \cosh 1 - i \sin 2 \sinh 1$; (2) $i \sinh 2$; (3) $\dfrac{\sin 4 - i \sinh 2}{2(\cos^2 2 + \sinh^2 1)}$;

(4) $\dfrac{8+15i}{17}$; (5) $\dfrac{\sinh 8 - i \sin 4}{2(\cosh^2 4 - \cos^2 2)}$; (6) i.

67. $\operatorname{Im} e^z = 0$, if $\operatorname{Im} z = k\pi$; $\operatorname{Re} e^z = 0$, if $\operatorname{Im} z = (2k+1)\pi/2$;
$\operatorname{Im} \cos z = 0$, if $\operatorname{Re} z = k\pi$ or $\operatorname{Im} z = 0$;
$\operatorname{Re} \cos z = 0$, if $\operatorname{Re} z = (2k+1)\pi/2$;

$\operatorname{Im} \sin z = 0$, if $\operatorname{Re} z = (2k+1)\dfrac{\pi}{2}$ or $\operatorname{Im} z = 0$;

$\operatorname{Re} \sin z = 0$, if $\operatorname{Re} z = k\pi$; $\operatorname{Im} \tan z = 0$, if $\operatorname{Im} z = 0$;

$\operatorname{Re} \tan z = 0$, if $\operatorname{Re} z = \dfrac{k\pi}{2}$; $\operatorname{Im} \cosh z = 0$, if $\operatorname{Im} z = k\pi$, or $\operatorname{Re} z = 0$;

$\operatorname{Re} \cosh z = 0$, if $\operatorname{Im} z = (2k+1)\dfrac{\pi}{2}$;

$\operatorname{Im} \coth z = 0$, if $\operatorname{Im} z = \dfrac{k\pi}{2}$; $\operatorname{Re} \coth z = 0$, if $\operatorname{Re} z = 0$.
In each case k is an integer ($k = 0, \pm 1, \pm 2, \ldots$).

68. (1) $\log 4 + 2k\pi i$, $(2k+1)\pi i$, πi; (2) $\left(2k+\dfrac{1}{2}\right)\pi i$, $\dfrac{\pi i}{2}$; (3) $\left(2k \pm \dfrac{1}{4}\right)\pi i$;

(4) $\dfrac{1}{2}\log 13 + \left(2k\pi - \tan^{-1}\dfrac{3}{2}\right)i$, $\dfrac{1}{2}\log 13 + \left[(2k+1)\pi - \tan^{-1}\dfrac{3}{2}\right]i$.

69. The set of values of $2\operatorname{Log} z$ comprises only part of the set of values of $\operatorname{Log}(z^2)$ (see [2, Chapter III, sec. 19]).

70. (1) $4\pi i$; (2) $-2\pi i$; (3) 0; (4) $4\pi i$.

71. (1) $\cos(2k\sqrt{2}.\pi) + i \sin(2k\pi\sqrt{2})$;

(2) $2^{\sqrt{2}}[\cos(2k+1)\pi\sqrt{2} + i \sin(2k+1)\pi\sqrt{2}$;

(3) $e^{2k\pi}(\cos \log 2 + \sin \log 2)$;

(4) $e^{2k\pi}$; (5) $e^{\left(2k-\frac{1}{2}\right)\pi}$; (6) $\dfrac{1-i}{\sqrt{2}} e^{\left(2k+\frac{1}{4}\right)\pi}$;

(7) $5e^{\tan^{-1}\frac{4}{3}+2k\pi}\left[\cos\left(\log 5 - \tan^{-1}\dfrac{4}{3}\right) + i \sin\left(\log 5 - \tan^{-1}\dfrac{4}{3}\right)\right]$;

(8) $-5e^{\tan^{-1}\frac{4}{3}+(2k+1)\pi}\left[\cos\left(\log 5 - \tan^{-1}\dfrac{4}{3}\right) + i \sin\left(\log 5 - \tan^{-1}\dfrac{4}{3}\right)\right]$.

Everywhere k is an integer ($k = 0, \pm 1, \pm 2, \ldots$).

73. The sets of values of $a^{2\alpha}$ and $(a^\alpha)^2$ are identical, but generally speaking are not identical with the set of values of $(a^2)^\alpha$; the general case when the

index 2 is replaced by the arbitrary complex number β, is considered in [2, Chapter III, sec. 20].

77. (1) $\operatorname{Im}\cos^{-1}z = \operatorname{Im}\sin^{-1}z = 0$, if z is a real number and $|z| \leqslant 1$; $\operatorname{Im}\tan^{-1}z = 0$, if z is a real number;

(2) $\operatorname{Re}\sinh^{-1}z = 0$, if z is a purely imaginary number and $|z| \leqslant 1$.

78. (1) $\frac{\pi}{6}+2k\pi$, $\frac{5\pi}{6}+2k\pi$; (2) $2k\pi\pm\frac{\pi}{3}$;

(3) $2k\pi\pm i\log(2+\sqrt{3})$; (4) $2k\pi-i\log(\sqrt{2}-1)$, $(2k+1)\pi-i\log(\sqrt{2}+1)$;

(5) $\frac{1}{2}\left[\tan^{-1}\frac{1}{2}+(2k+1)\pi\right]+\frac{i}{4}\log 5$; (6) $\log(\sqrt{5}\pm 2)+\left(2k\pm\frac{1}{2}\right)\pi i$;

(7) $\frac{1}{4}\log 5+\left[\frac{1}{2}\tan^{-1}2+\left(k+\frac{1}{2}\right)\pi\right]i$. Everywhere k is an integer ($k = 0, \pm 1, \pm 2, \ldots$).

79. The straight line segment: $x = 1$, $-2 \leqslant y \leqslant 0$.

80. The parabola $y = x^2$.

81. The right-hand half of the parabola $y = x^2$, traversed twice.

82. A left semicircle of radius a with centre at the point $z = 0$.

83. The branch of the hyperbola $y = 1/x$, situated in the third quadrant.

84. (1) The upper semicircle of radius 1 with centre at the point $z = 0$;

(2) The quarter circumference of radius 1 with centre at the point $z = 0$, situated in the first quadrant.

85. (1) The cycloid: $x = a(t-\sin t)$, $y = a(1-\cos t)$; (2) The first (counting from the coordinate origin) arc of the curtate ($a < b$), prolate ($a > b$) or ordinary ($a = b$) cycloid: $x = at-b\sin t$, $y = a-b\cos t$.

86. (1) The images of the straight lines $x = C$ are for $C \neq 0$ the parabolas $u = C^2-\frac{v^2}{4C^2}$, for $C = 0$ the semi-axis $v = 0$, $u \leqslant 0$; the images of the straight lines $y = C$ are for $C \neq 0$ the parabolas $u = \frac{v^2}{4C^2}-C^2$, for $C = 0$ the semi-axis $v = 0$, $u \geqslant 0$; the image of the straight line $y = x$ is the semi-axis $u = 0$, $v \geqslant 0$; the images of the circles $|z| = R$ are the circles $|w| = R^2$; the images of the rays $\arg z = \alpha$ are the rays $\arg w = 2\alpha$; the straight lines $x = C$, $y = C$ for $C \neq 0$ and the rays $\arg z = \alpha$ are mapped one-one; (2) The originals of the straight lines $u = C$ are the hyperbolas $x^2-y^2 = C$ (for $C = 0$, a pair of straight lines), the originals of the straight lines $v = C$ are the hyperbolas $xy = \frac{C}{2}$ (for $C = 0$, a pair of straight lines).

87. (1) The images of the straight lines $x = C$ are the circles $u^2+v^2-\frac{u}{C} = 0$, for $C = 0$, the axis $u = 0$; the images of the straight lines $y = C$ are the circles $u^2+v^2+\frac{v}{C} = 0$, for $C = 0$ the axis $v = 0$; the images of the circles $|z| = R$ are the circles $|w| = 1/R$; the images of the rays $\arg z = \alpha$ are the rays $\arg z = -\alpha$; the image of the circle $|z-1| = 1$ is the straight line $u = \frac{1}{2}$;

(2) The originals of the straight lines $u=C$ are the circles $x^2+y^2-\frac{x}{C}=0$, for $C=0$, the axis $x=0$; the originals of the straight lines $v=C$ are the circles $x^2+y^2+\frac{y}{C}=0$, for $C=0$, the axis $y=0$.

88. The function $w=z+\frac{1}{z}$ maps the circles $|z|=R\neq 1$ onto the ellipses $\frac{u^2}{\left(R+\frac{1}{R}\right)^2}+\frac{v^2}{\left(R-\frac{1}{R}\right)^2}=1$, and the circle $|z|=1$ onto the segment $v=0$, $-2\leqslant u\leqslant 2$; the function $w=z-\frac{1}{z}$ maps the circles $|z|=R\neq 1$ onto the ellipses $\frac{u^2}{\left(R-\frac{1}{R}\right)^2}+\frac{v^2}{\left(R+\frac{1}{R}\right)^2}=1$, and the circle $|z|=1$ onto the segment $u=0$, $-2\leqslant v\leqslant 2$.

89. The original of the family $u=C$ is the family $x(x^2+y^2+1)=C(x^2+y^2)$; the original of the family $v=C$ is the family $y(x^2+y^2-1)=C(x^2+y^2)$ (Fig. 58).

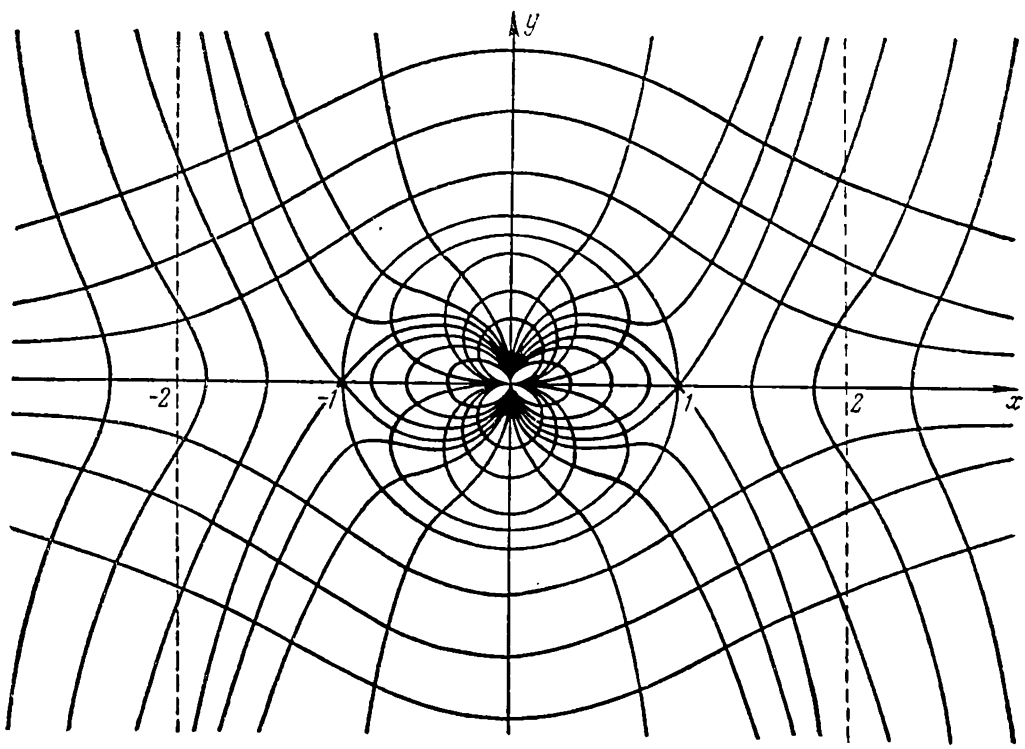

Fig. 58

90. On the ray going along the negative part of the real axis from the point $w = -\frac{1}{4}$ to the point $w = \infty$.

91. (1) The circles $\varrho = e^C$, the rays $\theta = C$, the logarithmic spiral $\varrho = e^\theta$; (2) The curves $y = e^x + 2k\pi$.

92. (1) The family of straight lines $x = C$ is transformed into the family $v^2 = -4a^2\left(u - a^2 + \frac{1}{4}\right)$ (parabolas with foci at the point $w = -\frac{1}{4}$, $a = C + \frac{1}{2}$), to which also belongs the ray going from the point $w = -\frac{1}{4}$ along the negative part of the real axis ($a = 0$); the family $y = C$ is transformed into the family of confocal parabolas $v^2 = 4C^2\left(u + C^2 + \frac{1}{4}\right)$, to which also belongs the ray going from the point $w = -\frac{1}{4}$ along the real axis in the positive direction;

(2) The family $x = C$ is transformed into a family of circles of Apollonius with respect to the points $w = -1$ and $w = 1$ (including also the imaginary axis); the equation of the family of circles of Apollonius is: $(u - a)^2 + v^2 = a^2 - 1$; $|a| > 1$ ($a = \coth 2C$); the family $y = C$ is transformed into a family of circular arcs with ends at the points $w = -1$ and $w = 1$, including also the corresponding parts of the real axis; the equation of the pencil of circles is $u^2 + (v + b)^2 = 1 + b^2$ ($b = \cot 2C$);

(3) The family $x = C$ is transformed into the family of spirals $\varrho = e^{C^2 - \frac{\theta^2}{4C^2}}$, the axis $x = 0$ corresponding to the segment $\theta = 0$, $0 < \varrho \leqslant 1$; the family $y = C$ is transformed into the family of spirals $\varrho = e^{\frac{\theta^2}{4C^2} - C^2}$, the axis $y = 0$ corresponding to the ray $\theta = 0$, $1 \leqslant \varrho < \infty$.

93. To the straight lines $y = C$ correspond the curves $u = x + e^x \cos C$, $v = C + e^x \sin C$, to the rectilinear segments $x = C$ there correspond arcs of the curves $u = C + e^C \cos y$, $v = y + e^C \sin y$ (Fig. 59).

94. (1) To the family $|w| = R$ there correspond the circles $r = (\cos\phi)/\log R$ ($R \neq 1$) and the imaginary axis ($R = 1$); to each ray $\arg w = \alpha$ there corresponds a family of circles $r = \frac{\sin\phi}{2k\pi - \alpha}$; for $\alpha = 0$ the real axis belongs to this family (for $k = 0$); (2) The hyperbolas $x^2 - y^2 = \log R$ and $2xy = \alpha + 2k\pi$.

95. (1) $z = 0$ and $z = 2$; (2) $z = 0$, $z = 1/m$, $z = i/n$ (m and n are any integers); (3) All the points of the plane; (4) All the points of the disk $|z| \leqslant 1$.

97. (2) When $\lim_{n\to\infty} z_n$ equals 0 or ∞.

100. Only $f(z) = \frac{z \operatorname{Re} z}{|z|}$ ($f(0) = 0$).

101. (1) and (2). Continuous but not uniformly.

102. (2) No; (3) Yes.

106. $r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \phi}$, $\frac{\partial u}{\partial \phi} = -r\frac{\partial v}{\partial r}$.

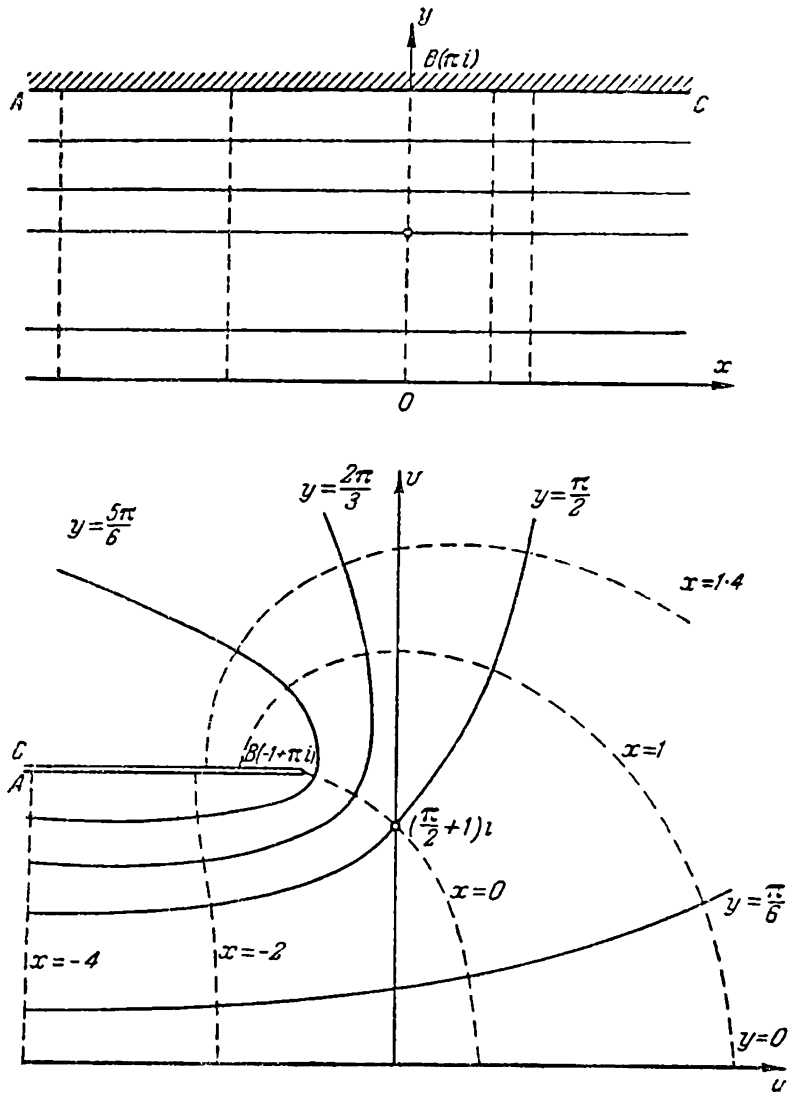

FIG. 59

109. 0.

118. (1) No, if $u \neq \text{const}$; (2) $f(u) = au+b$.

119. $|f(z)|$ is not a harmonic function, $\arg f(z)$ and $\log |f(z)|$ are harmonic.

120. $\Delta u = \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \phi^2}$; $u = C_1 \log r + C_2$.

121. $p_1 = x$, $q_1 = y$; $p_2 = x^2-y^2$, $q_2 = 2xy$; $p_3 = x^3-3xy^2$, $q_3 = 3x^2y - y^3$; $p_4 = x^4-6x^2y^2+y^4$; $q_4 = 4x^3y-4xy^3$; $p_n = r^n \cos n\phi$, $q_n = r^n \sin n\phi$.

122. $v(x, y) = 2xy+y+C$.

123. $v(x, y) = -\dfrac{y}{x^2+y^2} + C$.

124. (a) $v(x, y) = \arg z + C$; (b) $v(x, y) = \arg z + 2m\pi + C$ (the expression for $\arg z$ in terms of x and y is given in problem 2, (10)).

125. (a) $v(x, y) = \arg z - \arg (z-1) + 2m\pi + C$; (b) $v(x, y) = \arg z - \arg (z-1) + C$; (c) $v(x, y) = \arg z - \arg (z-1) + 2m\pi + C$.

126. (a) $v(x, y) = \sum\limits_{k=1}^{n} \alpha_k \arg (z-z_k) + 2\pi \sum\limits_{k=1}^{n} m_k \alpha_k + C$;

(b) $v(x, y) = \sum\limits_{k=1}^{n} \alpha_k \arg (z-z_k) + 2\pi m \sum\limits_{k=1}^{n} \alpha_k + C$ $\left(\text{if } \sum\limits_{k=1}^{n} \alpha_k = 0 \text{ the function } v(x, y) \text{ is single valued in the given domain!}\right)$.

127. (1) It exists; (2) It exists; (3) It does not exist.

128. $f(z) = z^2 + (5-i)z - \dfrac{i}{z} + Ci$.

129. $f(z) = ze^z + 2i \cos z + z^3 - iz + Ci$.

130. $f(z) = \dfrac{1}{2z} + iz^2 + 3i + C$.

131. $f(z) = 2i \log z - (2-i)z + C$. Everywhere C is an arbitrary real constant.

132. $u = C_1 x + C_2$.

133. $u = C_1(ax+by) + C_2$.

134. $u = C_1 \tan^{-1}\dfrac{y}{x} + C_2$.

135. $u = C_1 xy + C_2$.

136. $u = C_1 \log (x^2+y^2) + C_2$.

137. $u = \dfrac{C_1 x}{x^2+y^2} + C_2$.

138. $u = C_1 \sqrt{[x+\sqrt{(x^2+y^2)}]} + C_2$.

139. It does not exist.

140. $f(z) = e^{i\alpha} z^2 e^z$.

141. $f(z) = e^{i\alpha} e^{z^2}$.

142. $f(z) = A e^{\frac{z^2}{2}}$.

143. $f(z) = Aze^z$ (α is an arbitrary real constant, A is an arbitrary positive constant).

145. $az+\lambda$, $iaz+\lambda$, λe^{az}, λe^{iaz}.

146. $iaz+\lambda$, $az+\lambda$, λe^{iaz}, λe^{az}.

147. $ai \log z+\lambda$, $a \log z+\lambda$, $\lambda e^{ai \log z}$, $\lambda e^{a \log z}$.

148. $a \log z+\lambda$, $ai \log z+\lambda$, $\lambda e^{a \log z}$, $\lambda e^{ai \log z}$.

149. $\frac{a}{z}+\lambda$, $\frac{ai}{z}+\lambda$, $\lambda e^{\frac{a}{z}}$, $\lambda e^{\frac{ai}{z}}$ (a is an arbitrary real constant, λ is an arbitrary complex constant).

150. For $w = z^2$: (1) $\theta = 0, k = 2$; (2) $\theta = \pi, k = \frac{1}{2}$; (3) $\theta = \pi/4$, $k = 2\sqrt{2}$; (4) $\theta = \pi - \tan^{-1}\frac{4}{3}$, $k = 10$.

For $w = z^3$: (1) $\theta = 0$, $k = 3$; (2) $\theta = 0$, $k = 3/16$; (3) $\theta = \pi/2$, $k = 6$; (4) $\theta = -2 \tan^{-1}\frac{4}{3}$, $k = 75$.

151. (1) A compression for $|z| < \frac{1}{2}$, a stretch for $|z| > \frac{1}{2}$;

(2) A compression for $|z+1| < \frac{1}{2}$, a stretch for $|z+1| > \frac{1}{2}$;

(3) A compression for $|z| > 1$, a stretch for $|z| < 1$;
(4) A compression for $\operatorname{Re} z < 0$, a stretch for $\operatorname{Re} z > 0$;
(5) A compression for $|z-1| > 1$, a stretch for $|z-1| < 1$.

152. $S = \iint_G |f'(z)|^2 \, dx\, dy$, $L = \int_l |f'(z)| \, ds$.

153. $\sqrt{2}(e^{2\pi}-1)$.

154. $2e^2(e^2-1)$.

155. The domain D is the ring $e \leqslant |w| \leqslant e^2$. It is not possible to apply the formula of problem **152** since the mapping is not one-one.

CHAPTER II

156. $w = (1+i)(1-z)$.

157. $w = (2+i)z+1-3i$.

158. (1) $z_0 = -1+3i$, $\theta = 0$, $k = 2$, $w+1-3i = 2(z+1-3i)$;
(2) $z_0 = 2+2i$, $\theta = \pi/2$, $k = 1$, $w-2-2i = i(z-2-2i)$;
(3) there is no finite fixed point;
(4) if $a = 1$ there is no finite fixed point; if $a \neq 1$, then

$$z_0 = \frac{w_1-az_1}{1-a},\quad \theta = \arg a,\ k = |a|,\ w-\frac{w_1-az_1}{1-a} = a\left(z-\frac{w_1-az_1}{1-a}\right),$$

(5) If $a = 1$ there is no finite fixed point; if $a \neq 1$, then $z_0 = \dfrac{b}{1-a}$,

$$\theta = \arg a,\ k = |a|,\ w-\frac{b}{1-a} = a\left(z-\frac{b}{1-a}\right).$$

159. (1) $w = az+b$; (2) $w = -az+b$; (3) $w = -i(az+b)$; (4) $w = az+bi$. Everywhere a and b are real numbers and $a > 0$.

160. (1) $w = z+bi$ or $w = -z+1+bi$;
(2) $w = z+b$ or $w = -z-i+b$;
(3) $w = z+b(1+i)$ or $w = -z+1+b(1+i)$.
Everywhere b is a real number. It is possible for correspondences to exist between points lying either on a straight line parallel to the boundaries of the strip, or on parallel straight lines symmetrical with respect to the centre line of the strip. The mapping is not uniquely determined if the points lie on the centre line of the strip.

161. (1) $w = (z-a)/h$; (2) $w = \dfrac{-z+a+h}{h}+i$;

(3) $w = \dfrac{\sqrt{(1+k^2)}}{b}\,e^{-i\left(\frac{\pi}{2}+\tan^{-1}k\right)}z$,

(4) $w = \dfrac{\sqrt{(1+k^2)}}{b_2-b_1}\,e^{-i\left(\frac{\pi}{2}+\tan^{-1}k\right)}(z-ib_1)$.

162. $w = e^{i\alpha}Rz+w_0$.

163. (1) The family of straight lines $u = 1/a$ parallel to the imaginary axis (not including the imaginary axis itself);
(2) The family of straight lines $v = -1/b$, parallel to the real axis (not including the real axis itself);
(3) The family of circles $b(u^2+v^2)+u+v = 0$, touching the straight line $v = -u$ at the origin (also including this straight line);
(4) The pencil of straight lines $v = -ku$;
(5) The family of eircles passing through the coordinate origin and through the point $w_0 = 1/z_0$ (the straight line which passes through the points $w = 0$ and $w = w_0$ also belongs to this pencil);
(6) The eissoid $u^2 = -v^3/(v+1)$.

164. (1) Into a family of circles touching at the point $w = h$ the straight lines, respectively parallel to the imaginary and real axes (including also these straight lines); the equations of these families are;

$$(C-x_0)[(u-h_1)^2+(v-h_2)^2]-(u-h_1) = 0;$$

$$(C-y_0)[(u-h_1)^2+(v-h_2)^2]+(v-h_2) = 0,$$

where $z_0 = x_0+iy_0$, $h = h_1+ih_2$;

(2) Into the family of circles with centre at the point $w = h\left(|w-h| = \frac{1}{R}\right)$ and a family of rays issuing from the point $w = h$ ($\arg(w-h) = -\alpha$).

165. (1) The equation of a family of circles of Apollonius with respect to the points z_1 and z_2: $\left|\frac{z-z_1}{z-z_2}\right| = \lambda$. The ends A and B of the diameter lying on the straight line passing through the points z_1 and z_2 (Fig. 60), divide the

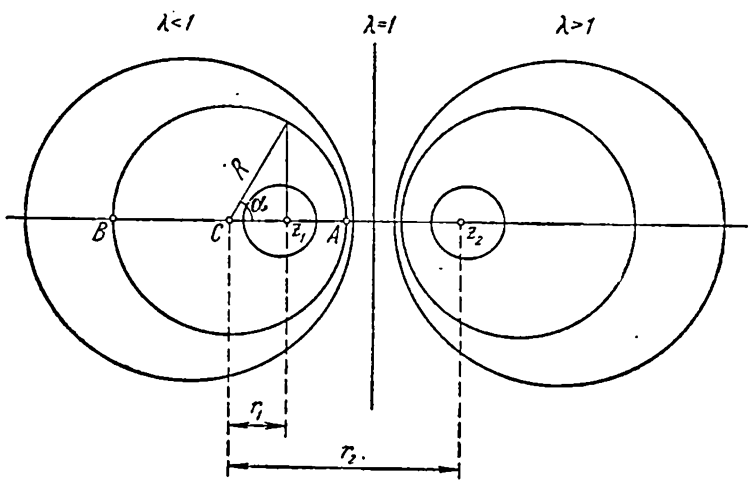

FIG. 60

segment $z_1 z_2$ internally and externally in the ratio λ. If use is made of the notation indicated in the diagram (C is the centre of the circle with diameter AB), and we put $|z_1-z_2| = d$ then for $\lambda < 1$ we have the relations:

$$\lambda = \cos\alpha = \frac{r_1}{R} = \frac{R}{r_2},\quad R = \frac{\lambda d}{1-\lambda^2},\quad r_1 = \frac{\lambda^2 d}{1-\lambda^2},\quad r_2 = \frac{d}{1-\lambda^2};$$

(2) Circular arcs passing through the points z_1 and z_2: $\arg\frac{z-z_1}{z-z_2} = \theta$. The arcs corresponding to the values $\theta = \alpha$ and $\theta = \pi-\alpha$ are complements and form a complete circle;

(3) To the polar net there corresponds (Fig. 61) a net consisting of the

circles of Apollonius $\left|\frac{z-z_1}{z-z_2}\right| = R$ and the arcs $\arg\frac{z-z_1}{z-z_2} = \theta$ orthogonal to them (if $\theta > 0$, then the arcs are situated on the right of the direction $\overrightarrow{z_1z_2}$; if $\theta < 0$, they are on the left of it);

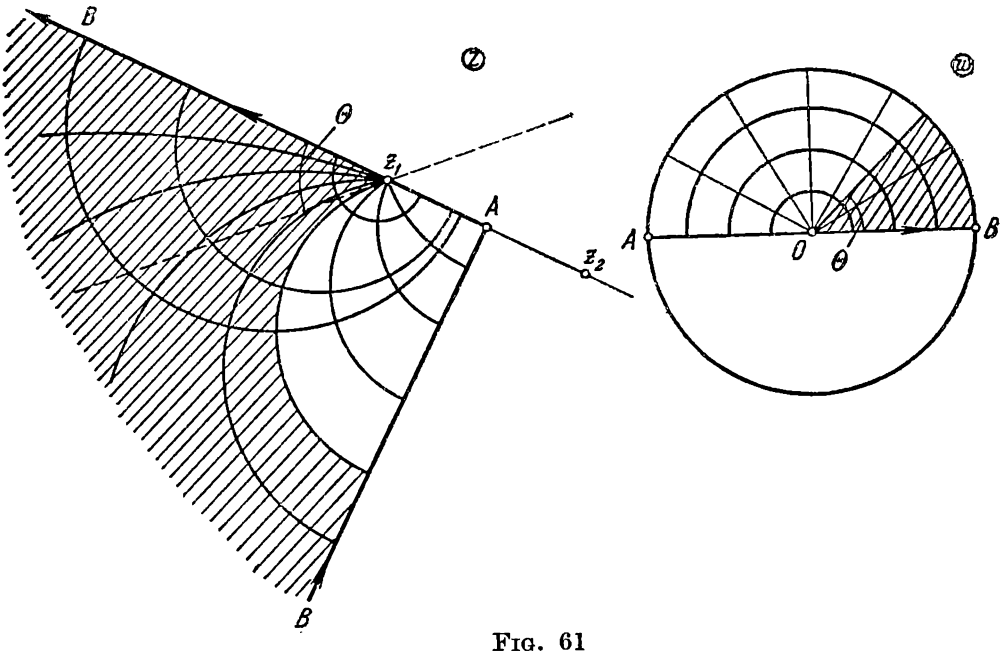

FIG. 61

(4) to the upper semicircle there corresponds the right angle indicated in the figure.

166. The semicircle $|w| < 1$, $\operatorname{Im} w < 0$.

167. The domain containing the point $w = 0$ and bounded by arcs of the circles $|w| = 1$ and $\left|w + \frac{5i}{4}\right| = 3/4$.

168. The domain obtained from the lower half-plane ($\operatorname{Im} w < 0$) by the removal of the points of the disk $\left|w - \frac{1}{2} + \frac{i}{2}\right| < \frac{\sqrt{2}}{2}$ which are in this half-plane.

169. (1) The domain bounded by the straight line $\operatorname{Re} w = 1$ and the circle $\left|w - \frac{1}{2}\right| = \frac{1}{2}$ tangential to it;

(2) The domain bounded by the tangential circles $\left|w - \frac{1}{2}\right| = \frac{1}{2}$ and $\left|w - \frac{3}{4}\right| = \frac{1}{4}$.

170. The doubly connected domain the boundary of which consists of the straight line $\operatorname{Re} w = \frac{1}{2}$ and the circle $\left|w - \frac{4}{3}\right| = \frac{2}{3}$.

171. (1) $w = -\frac{d}{z} + 1 + hi$ or $w = \frac{d}{z} + hi$;

(2) $w = \frac{d_1}{d_2 - d_1}\left(\frac{d_2}{z} - 1\right) + hi$ or $w = \frac{d_2}{d_1 - d_2}\left(\frac{d_2}{z} - 1\right) + 1 + hi$;

(3) $w = \frac{d_1(z - d_2)}{z(d_1 + d_2)}$.

172. (1) $w = -\frac{2i(z+1)}{4z - 1 - 5i}$; (2) $w = \frac{(1+2i)z + 6 - 3i}{5(z-i)}$.

173. (1) $w = \frac{(1+i)z + 1 + 3i}{(1+i)z + 3 + i}$; (2) $w = \frac{iz + 2 + i}{z+1}$; (3) $w = \frac{1-i}{2}(z+1)$.

174. (1) $w = \frac{(-1+3i)z + 1 - i}{(1+i)z - 1 + i}$; (2) $w = \frac{z(1-4i) - 2(1-i)}{2z(1-i) - (4-i)}$;

(3) $w = \frac{z(3-i) - (1+i)}{(1+i)(1-z)}$.

175. $w = \frac{z-i}{iz-1}$; the upper half-plane maps onto the unit disk.

176. (1) $w = (az+b)/(cz+d)$, where a,b,c,d are real numbers and $ad - bc > 0$;
(2) $w = (az+b)/(cz+d)$, where a,b,c,d are real numbers and $ad - bc < 0$;
(3) $w = i(az+b)/(cz+d)$, where a,b,c,d are real numbers and $ad - bc < 0$.

177. (1) $w = \frac{2}{2-z}$; (2) $w = -2\left(\frac{2z+1}{z-2}\right)$.

178. $w = (R-z)/(R+z)$; the image of the upper semicircle is the quadrant $u > 0, v < 0$.

179. (1) $\frac{2+i}{5}$; (2) $\frac{9}{2} + i$.

180. (1) $|z| = 2$; (2) the straight line $x = \frac{1}{2}$; (3) $\left|z - \frac{i}{4}\right| = \frac{1}{4}$;

(4) $xx_0 + yy_0 = \frac{1}{2}$; (5) $|z - z_0| = \sqrt{(|z_0|^2 - 1)}$ (that is, this circle is symmetrical to itself with respect to the unit circle);
(6) $(x^2+y^2)^2 - (x^2 - y^2) = 0$ (a lemniscate); (7) a curvilinear triangle with vertices at the points $\frac{1}{\overline{z}_1}$, $\frac{1}{\overline{z}_2}$, $\frac{1}{\overline{z}_3}$, and with sides which are arcs of circles passing through the point $z = 0$ (one of the arcs may be a segment of a straight line).

182. (1) $\theta(x) = \alpha + 2 \arg(x - \beta)$; (2) $w'(\beta) = e^{\left(\alpha - \frac{\pi}{2}\right)i} \Big/ 2b$

(3) If $b \geqslant 2$, then the whole half-plane is compressed, if $b < 2$, then the domain within the circle $|z-\overline{\beta}| < \sqrt{(2b)}$ is stretched. (The circle $|z-\overline{\beta}| = \sqrt{(2b)}$ is termed *isometric.*)

183. $w = (z-i)/(z+i)$; (2) $w = i\dfrac{z-2i}{z+2i}$; (3) $w = e^{i\left(\frac{\pi}{2}+\theta\right)}\dfrac{z-(a+bi)}{z-(a-bi)}$.

184. $w = Ri\dfrac{z-i}{z+i} + w_0$.

185. $w = -(z-2i)/(z+2i)$

186. $w = -4\dfrac{zi+2}{z-2-4i}$.

187. $w = k\,e^{\frac{i}{2}\left(\pi+\arg\frac{z_2}{z_1}\right)}\dfrac{z-z_1}{z-z_2}$, where k is a real number the sign of which is identical with the sign of $\arg z_2/z_1$. Rays issuing from the point $w = 0$ in the half-plane $\operatorname{Re} w > 0$ correspond to circular arcs, situated within the disk $|z| < 1$ and passing through the points z_1 and z_2 in the z-plane; the semicircles with centre at the point $w = 0$ in the half-plane $\operatorname{Re} w > 0$ correspond to arcs of the circles of Apollonius with respect to the points z_1 and z_2 situated within the disk $|z| < 1$.

188. $\dfrac{w-b}{w-\overline{b}} = e^{i\alpha}\dfrac{z-a}{z-\overline{a}}$.

189. $\dfrac{w-\overline{a}}{w-a} = i\dfrac{z-a}{z-\overline{a}}$.

190. (1) $\theta(\phi) = \alpha-\phi+2\arg(e^{i\phi}-a) = \alpha-\phi+2\tan^{-1}\dfrac{\sin\phi-\lambda\sin\theta}{\cos\phi-\lambda\cos\theta}$, where $a = \lambda e^{i\theta}$.

(2) $w'(0) = (1-|a|^2)e^{i\alpha}$, $\quad w'(a) = \dfrac{e^{i\alpha}}{1-|a|^2}$.

(3) If $a \neq 0$, then the domain lying inside the disk $\left|z-\dfrac{1}{\overline{a}}\right| < \sqrt{\left(\dfrac{1}{|a|^2}-1\right)}$ is stretched, and the domain outside this disk is compressed. (The circle $\left|z-\dfrac{1}{\overline{a}}\right| = \sqrt{\left(\dfrac{1}{|a|^2}-1\right)}$ is isometric.) If $a = 0$, then $|w'(z)| \equiv 1$.

(4) $\max\left|\dfrac{dw}{dz}\right| = \dfrac{1+|a|}{1-|a|}$, $\quad \min\left|\dfrac{dw}{dz}\right| = \dfrac{1-|a|}{1+|a|}$.

191. (1) $w = \dfrac{2z-1}{2-z}$; $\quad$ (2) $w = \dfrac{2iz+1}{2+iz}$; $\quad$ (3) $w = -iz$;

(4) $\dfrac{w-a}{1-\overline{a}w} = e^{i\alpha}\dfrac{z-a}{1-\overline{a}z}$.

192. $R_2\dfrac{w-b}{R_2^2-\overline{b}w} = e^{i\alpha}R_1\dfrac{z-a}{R_1^2-\overline{a}z}$.

193. $w = (1-z)/(z+2)$

194. $w = 2\dfrac{z-2+i}{iz+2-2i}$.

195. (1) $w = R^2 e^{i\alpha}\dfrac{z-a}{R^2-\bar{a}z}$; (2) $\dfrac{w-b}{R^2-\bar{b}w} = e^{i\alpha}\dfrac{z-a}{R^2-\bar{a}z}$; (3) $w = R^2\dfrac{z-a}{R^2-az}$, where a is a real number and $|a| < 1$.

196. $w = \dfrac{(1+k)z-(z_2+kz_1)}{(1-k)z-(z_2-kz_1)}a$, where $k = \sqrt{\left(\dfrac{1-|z_2|^2}{1-|z_1|^2}\right)}e^{i\arg(1-\bar{z}_1z_2)}$,

$|a| = \dfrac{|z_1-z_2|}{|1-\bar{z}_1z_2|+\sqrt{[(1-|z_1|^2)(1-|z_2|^2)]}}$.

197. $w = \pm\dfrac{az-1+\sqrt{(1-a^2)}}{[1-\sqrt{(1-a^2)}]z-a}$, $\varrho = 2\dfrac{1-\sqrt{(1-a^2)}}{a}$.

200. (1) $w = z$; (2) $w = \dfrac{z+2-\sqrt{3}}{1+(2-\sqrt{3})z}$; (3) $w = i\dfrac{z-2+\sqrt{3}}{1-(2-\sqrt{3})z}$.

203. Circles passing through the point z_0 and having at this point a tangent determined by the vector $\bar{h}$.

204. $w = \dfrac{(R-ki)z-R^2}{z-(R+ki)}$, where k is a real number, $k \neq 0$ (for $k = \infty$ it is necessary to put $w = z$).

209. If $|a| < \sin\dfrac{\lambda}{2}$, then the transformation is elliptic. If we use the notation: $|a| = \sin\dfrac{\lambda}{2}\sin\beta$, $z_1 = e^{i\left(\alpha+\frac{\lambda}{2}-\frac{\pi}{2}\right)}\tan\dfrac{\beta}{2}$, $z_2 = e^{i\left(\alpha+\frac{\lambda}{2}-\frac{\pi}{2}\right)}\cot\dfrac{\beta}{2}$, $\tan\dfrac{\lambda_0}{2} = \tan\dfrac{\lambda}{2}\cos\beta$, then the transformation can be written in the form $\dfrac{w-z_1}{w-z_2} = e^{i\lambda_0}\dfrac{z-z_1}{z-z_2}$. If $|a| = \sin\dfrac{\lambda}{2}$, then the transformation is parabolic and is of the form $\dfrac{1}{w-z_0} = \dfrac{1}{z-z_0}+ih\bar{z}_0$ where $z_0 = e^{i\left(\alpha+\frac{\lambda}{2}-\frac{\pi}{2}\right)}$, $h = \tan\dfrac{\lambda}{2}$. If $|a| > \sin\dfrac{\lambda}{2}$, then the transformation is hyperbolic. If we use the notation: $\sin\dfrac{\lambda}{2} = |a|\sin\beta$, $z_1 = e^{i\left(\alpha+\frac{\lambda}{2}-\beta\right)}$, $z_2 = e^{i\left(\alpha+\frac{\lambda}{2}-\pi+\beta\right)}$,

$K = \dfrac{\cos\dfrac{\lambda}{2}+\sqrt{\left(|a|^2-\sin^2\dfrac{\lambda}{2}\right)}}{\cos\dfrac{\lambda}{2}-\sqrt{\left(|a|^2-\sin^2\dfrac{\lambda}{2}\right)}}$, then the transformation is written in the form $\dfrac{w-z_1}{w-z_2} = K\dfrac{z-z_1}{z-z_2}$.

210. $\Gamma = 2\tan^{-1}\frac{a}{h}$; $\Gamma = \frac{2a}{h} + O\left[\left(\frac{a}{h}\right)^3\right]$ for small $\frac{a}{h}$;

$\Gamma = \pi - \frac{2h}{a} + O\left[\left(\frac{h}{a}\right)^3\right]$ for small $\frac{h}{a}$.

211. $\Gamma = \gamma + 2\tan^{-1}\frac{x_0\sin\gamma}{1-x_0\cos\gamma} = 2\tan^{-1}\left(\frac{1+x_0}{1-x_0}\tan\frac{\gamma}{2}\right)$;

$\Gamma < \gamma$, if $x_0 < 0$, and $\Gamma > \gamma$, if $x_0 > 0$.

215. (1) $w = -20/z$; (2) $w = -(2iz+1+2i)$.

216. $w = \frac{z - \sqrt{(h^2-R^2)}}{z+\sqrt{(h^2-R^2)}}e^{i\alpha}$, $\varrho = \frac{h}{R} + \sqrt{\left[\left(\frac{h}{R}\right)^2 - 1\right]}$.

217. $h = 5/4$; $w = 2e^{i\alpha}\frac{4z-3}{4z+3}$ or $w = e^{i\alpha}\frac{4z+3}{4z-3}$.

218. $w = \frac{2z}{z+24}e^{i\alpha}$ or $w = \frac{z+24}{3z}e^{i\alpha}$, $\varrho = \frac{2}{3}$.

219. $w = \lambda\frac{z-z_1'}{z-z_2'}$ or $w = \lambda\frac{z-z_2'}{z-z_1'}$ where λ is an arbitrary complex number,

$$z_1' = z_1 + \frac{u_1(z_2-z_1)}{d},\quad z_2' = z_1 + \frac{u_2(z_2-z_1)}{d},\quad d = |z_2-z_1|,$$

$$u_1 = \frac{1}{2d}(r_1^2+d^2-r_2^2 - \sqrt{([d^2-(r_1+r_2)^2]\,[d^2-(r_1-r_2)^2])},$$

$$u_2 = \frac{1}{2d}(r_1^2+d^2-r_2^2 + \sqrt{([d^2-(r_1+r_2)^2]\,[d^2-(r_1-r_2)^2])};$$

$$\mu\left(\text{or }\frac{1}{\mu}\right) = \left|\frac{(d+r_2-u_1)(r_1-u_2)}{(d+r_2-u_2)(r_1-u_1)}\right|.$$

220. (1) $\mu = 2$; (2) $\mu = 5+2\sqrt{6}$.

223. The group will be finite if a/π is rational.

225. The fundamental domains (some of their possible forms) are shown shaded in Fig. 62. Equivalent boundary sides are connected by arrows. Points with figures are fixed points of the rotations leading to the groups (the figures indicate the number of rotations). For the last five examples the parallelogram of the doubly periodic semigroup is indicated; in example (7) this is a square, in examples (8) and (9) it is a rhombus with angles 120° and 60°.

REMARK. It can be shown that apart from a linear transformation groups (3)–(9) exhaust the groups of linear transformations with one limit point (known as the limit point of a set of points equivalent to one another)†.

226. (1) $w = e^{i\alpha}z$; (2) $\frac{w-1}{w+1} = e^{i\alpha}\frac{z-1}{z+1}$; (3) $\frac{w-i}{w+i} = e^{i\alpha}\frac{z-i}{z+i}$;

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† See L. R. FORD, (1929), *Automorphic functions*, §§ 59,60, McGraw-Hill (reprint Chelsea, 1951).

(4) $\dfrac{w-a}{1+\bar{a}w} = e^{i\alpha}\dfrac{z-a}{1+\bar{a}z}$.

232. (1) and (2). The construction is obvious; (3) The equidistants of the "straight line" $\alpha\beta$ (α and β are the "infinitely distant" points of this straight line) are circular arcs with ends at α, β (known as *hypercycles*); (4) The limiting lines of a pencil of "parallel straight lines" with common point "at infinity" α are circles touching (internally) the unit circle at the point α (they are known as *oricycles*).

233. (1) For the construction of a "rectilinear" triangle with angles ϕ_1, ϕ_2, ϕ_3, construct a circular sector OAB with central angle $\Delta = \pi-(\phi_1+\phi_2+\phi_3)$ draw the "straight line" AB, and through the points A and B "straight lines" at angles ϕ_2 and ϕ_3 to AB, intersecting at the point C. The triangle ABC is the one required.

234. (1) $w = i(z^2-a^2)$; (2) $w = \sqrt{\left(z-\dfrac{p}{2}\right)-i\sqrt{(p/2)}}$.

235. (1) $w = z^2/a^2$; (2) $w = \sqrt{(z/a)}$; (3) $w = \dfrac{A}{2}\left(\dfrac{z+1}{z-2}\right)^2$.

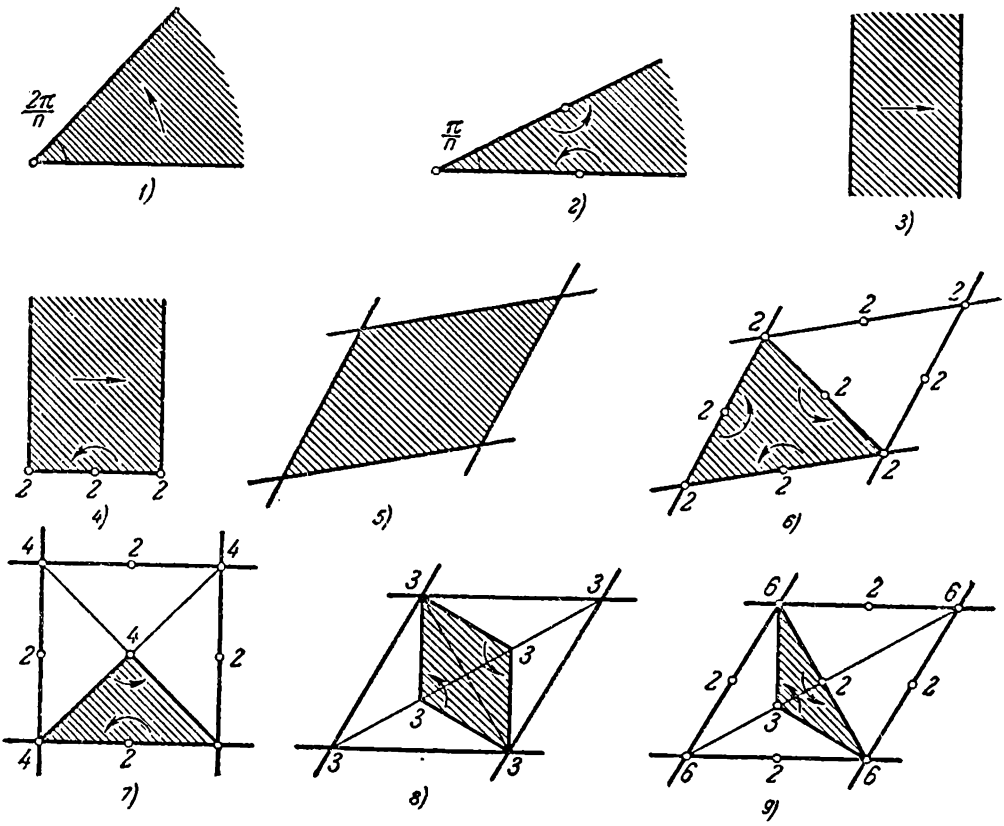

FIG. 62

236. The domain is bounded by Pascal's limaçon $u = R(\cos\phi + m\cos 2\phi)$, $v = R(\sin\phi + m\sin 2\phi)$. If the coordinate origin in the w-plane is transferred to the point $w = -Rm$, then we obtain the equation of the limaçon in the usual form (in polar coordinates): $\varrho = R(1+2m\cos\theta)$. For $m = 0$ Pascal's limaçon becomes a circle, for $m = 1/2$, a cardioid with cusp $w = -R/2$. The images of the circles $|z| = r < 1$ are also limaçons of Pascal the polar equations of which are easily obtained by transferring the coordinate origin to the point $w = -Rmr^2$: $\varrho = Rr(1+2mr\cos\theta)$. The images of the radii of the circle $\arg z = \alpha$ are parabolas passing through the coordinate origin: $m(u\sin 2\alpha - v\cos 2\alpha)^2 + R\sin\alpha(u\sin\alpha - v\cos\alpha) = 0$. To the radii $\alpha = 0$ and $\alpha = \pi$ correspond the segments of the real axis $0 \leqslant u \leqslant R(1+m)$ and $R(m-1) \leqslant u \leqslant 0$.

237. The domain bounded by the parabola $u = -v^2$ and the curve $\varrho = 2\cos(\theta/3)$, $|\theta| < 3\pi/4$ (Fig. 63).

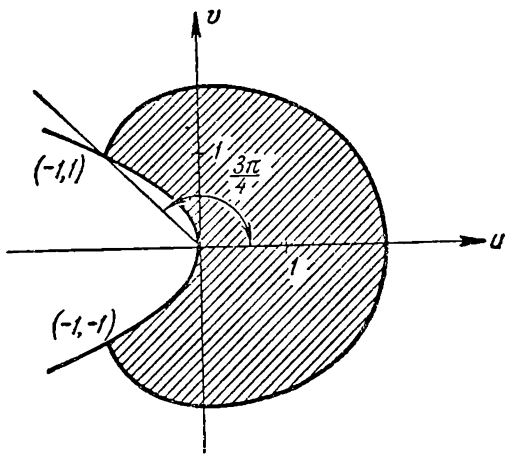

FIG. 63

238. (1) The domain bounded by the epicycloid $u = R\left(\cos\phi + \frac{\cos n\phi}{n}\right)$, $v = R\left(\sin\phi + \frac{\sin n\phi}{n}\right)$, having $(n-1)$ cusps which are the images of the points $z = \sqrt[n-1]{(-1)}$; (2) the exterior of the hypocycloid: $u = R\left(\cos\phi + \frac{\cos n\phi}{n}\right)$, $v = R\left(\sin\phi - \frac{\sin n\phi}{n}\right)$, having $(n+1)$ cusps (images of the points $z = \sqrt[n+1]{1}$).

239. (1) $|m| \leqslant 1/n$. The domain is bounded by an "elongated" epicycloid (epitrochoid), that is, the trajectery of a point at a distance mR from the centre of a circle of radius R/n, rolling outside a circle of radius $R(n-1)/n$.

(2) $|m| \leqslant 1/n$. In the first ease the exterior of the unit circle, and in the second case its interior is mapped onto the exterior of a "shortened" hypocycloid (hypotrochoid).

240. (1) $w = z^{\frac{1}{\alpha}}$; (2) $w = \dfrac{2\left(\sqrt[3]{(4)}+1\right)e^{\frac{\pi i}{3}} z^{\frac{4}{3}}}{\left(\sqrt[3]{(4)}-2\right)e^{\frac{\pi i}{3}} z^{\frac{4}{3}} + 3\sqrt[3]{4}}$.

241. (1) $w = \left(\dfrac{1+z}{1-z}\right)^2$; (2) $w = -\dfrac{z^2+1}{2z}$; (3) $w = \dfrac{-2z^2+3z-2}{2z^2+3z+2}$.

242. (1) $w = \dfrac{z^2+2iz+1}{iz^2+2z+i}$; (2) $w = \dfrac{2z^2+3iz+2}{2z^2-3iz+2}$.

243. $w = [(z-1)/(z+1)]^2$.

244. (1) $w = \left(\dfrac{z^{\frac{1}{\alpha}}+R^{\frac{1}{\alpha}}}{z^{\frac{1}{\alpha}}-R^{\frac{1}{\alpha}}}\right)$: (2) $w = \left(\dfrac{z^{\frac{1}{\alpha}}-R^{\frac{1}{\alpha}}}{z^{\frac{1}{\alpha}}+R^{\frac{1}{\alpha}}}\right)$.

245. $w = -\left(\dfrac{2z+\sqrt{(3)}-i}{2z-\sqrt{(3)}-i}\right)^{\frac{3}{2}}$.

246. $w = -\left(\dfrac{2z+\sqrt{(3)}-i}{2z-\sqrt{(3)}-i}\right)^3$.

247. $w = \left(\dfrac{2z+\sqrt{(3)}-i}{2z-\sqrt{(3)}-i}\right)^3$.

248. $w = i\left(\dfrac{2z+\sqrt{(3)}-i}{2z-\sqrt{(3)}-i}\right)^{\frac{3}{2}}$.

249. $w = \left[\dfrac{z-\sqrt{(2)}(1-i)}{z-\sqrt{(2)}(1+i)}\right]^4$.

250. $w = e^{\frac{\pi i}{3}}\left(\dfrac{z+1}{z-1}\right)^{\frac{2}{3}}$.

251. $w = \sqrt{[(z+1)/(1-z)]}$.

252. $w = \sqrt{[(z+i)/(i-z)]}$.

253. $w = \sqrt{[(z-z_1)/(z_2-z)]}$.

254. $w = \sqrt{[(z+R)/(z-R)]}$.

255. $w = e^{-\frac{\pi i}{8}}\sqrt{(z-i)}$.

256. $w = \sqrt{(z^2+h^2)}$.

257. $w = \dfrac{\sqrt{(z^2+h^2)}}{z}$.

258. $w = \left(\dfrac{\sqrt{(z)}+1}{\sqrt{(z)}-1}\right)^2$. For one choice of the branch of $\sqrt{z}$ this function gives the solution of problem 258, and for the other the solution of problem 259.

259. See the answer to problem 258.

260. $w = z/(1-z)^2$

261. To the circles $|z| = R$ (Fig. 64) there correspond the confocal ellipses $\dfrac{4u^2}{\left(R+\frac{1}{R}\right)^2}+\dfrac{4v^2}{\left(R-\frac{1}{R}\right)^2} = 1$ (to the circle $|z| = 1$ corresponds the segment $v = 0$, $-1 \leqslant u \leqslant 1$); to the rays $\arg z = \alpha$ there correspond branches of the confocal hyperbolas $u^2/\cos^2\alpha - v^2/\sin^2\alpha = 1$ (to the ray $\arg z = 0$ corresponds the ray $v = 0$, $u \geqslant 1$, to the ray $\arg z = \pi$, the ray $v = 0$, $u \leqslant -1$; to the rays $\arg z = \pm\dfrac{\pi}{2}$ corresponds the axis $u = 0$).

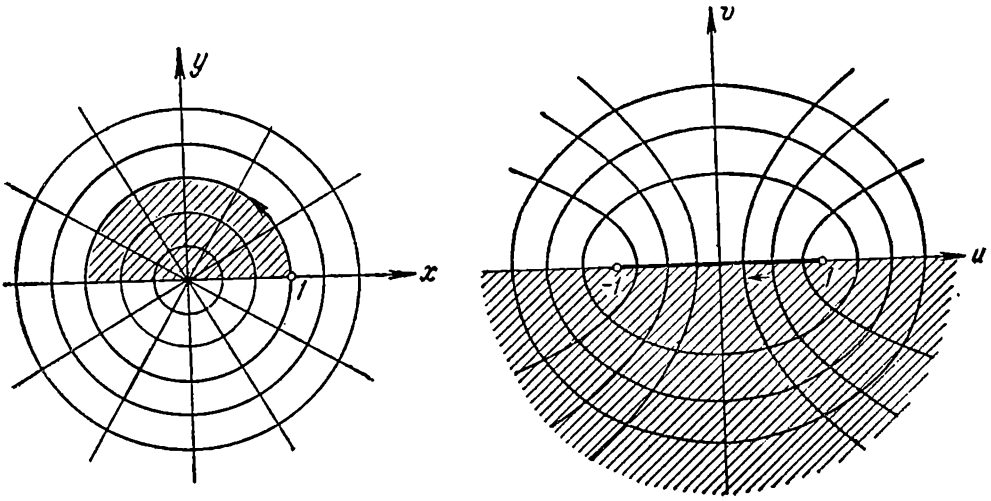

FIG. 64

262. (1) and (2). The exterior of the ellipse $\dfrac{4u^2}{\left(R+\dfrac{1}{R}\right)^2}+\dfrac{4v^2}{\left(R-\dfrac{1}{R}\right)^2}=1$ (Fig. 65, (1), (2));.

(3) and (4). The whole plane with a cut along the segment $[-1, 1]$ (Fig. 65, (3), (4));

(5) and (6). The whole plane with cuts along the rays $(-\infty, -1]$ and $[1, \infty]$, on the real axis;

(7) The lower half-plane;

(8) The upper half-plane;

(9) The upper half-plane;

(10) The upper half of the interior of the ellipse

$$\frac{4u^2}{\left(R+\dfrac{1}{R}\right)^2}+\frac{4v^2}{\left(R-\dfrac{1}{R}\right)^2}=1;$$

(11) The lower half of the interior of the ellipse

$$\frac{4u^2}{\left(R+\dfrac{1}{R}\right)^2}+\frac{4v^2}{\left(R-\dfrac{1}{R}\right)^2}=1;$$

(12) The right half of the interior of the ellipse

$$\frac{4u^2}{\left(R+\dfrac{1}{R}\right)^2}+\frac{4v^2}{\left(R-\dfrac{1}{R}\right)^2}=1$$

with a cut along the segment $\left[1, \dfrac{1}{2}\left(R+\dfrac{1}{R}\right)\right]$;

(13) the domain between the branches of the hyperbola $u^2/\sin^2\alpha - v^2/\cos^2\alpha = 1$

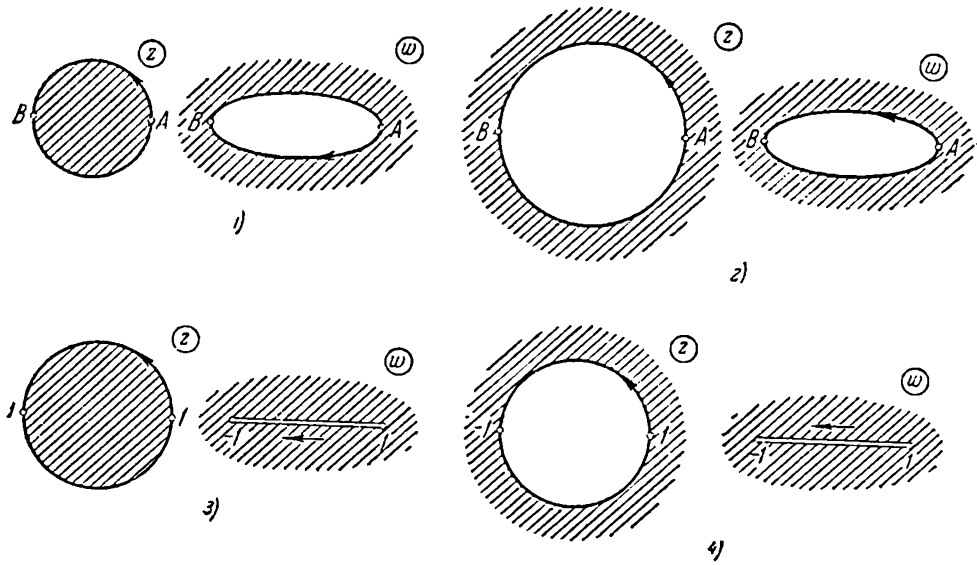

FIG. 65

263. (1) To the circles $|z| = R$ correspond the ellipses $\dfrac{4u^2}{\left(R-\dfrac{1}{R}\right)^2}+\dfrac{4v^2}{\left(R+\dfrac{1}{R}\right)^2} = 1$ (to the circle $|z| = 1$ corresponds the segment $u = 0$, $-1 \leqslant v \leqslant 1$), to the rays $\arg z = \alpha$ correspond the branches of the hyperbola $v^2/\sin^2\alpha - u^2/\cos^2\alpha = 1$ (to the rays $\phi = 0$ and $\phi = \pi$ corresponds the axis $v = 0$, to the ray $\phi = \pi/2$ corresponds the ray $u = 0$, $v \geqslant 1$, to the ray $\phi = -\pi/2$ corresponds the ray $u = 0$, $v \leqslant -1$);

(2) To the circles $|z| = R$ correspond the ellipses $\dfrac{4u^2}{\left(R+\dfrac{a^2}{R}\right)^2}+\dfrac{4v^2}{\left(R-\dfrac{a^2}{R}\right)^2} = 1$ (to the circle $|z| = a$ corresponds the segment $v = 0$, $-a \leqslant u \leqslant a$), to the rays $\arg z = \alpha$ correspond the branches of the hyperbola $\dfrac{u^2}{a^2\cos^2\alpha}-\dfrac{v^2}{a^2\sin^2\alpha} = 1$ (to the ray $\arg z = 0$ corresponds the ray $v = 0$, $u \geqslant a$, to the ray $\arg z = \pi$ corresponds the ray $v = 0$, $u \leqslant -a$, to the rays $\arg z = \pm\dfrac{\pi}{2}$ corresponds the axis $u = 0$);

(3) The families of confocal ellipses and hyperbolas obtained from the corresponding families of problem **261** by a rotation through an angle γ and a magnification with coefficient $|c|$ (the centre of the magnification is the coordinate origin).

264. (1) $w = \frac{e^{i\alpha}}{c}(z + \sqrt{(z^2 - c^2)})$;

(2) $w = \frac{1}{a+b}(z + \sqrt{[z^2 - (a^2 - b^2)]})$; in both cases for one choice of the branch of the root we obtain a mapping onto the exterior of the unit circle, and for the other onto the interior;

(3) $w = \frac{az - b\sqrt{[z^2 - (a^2 - b^2)]}}{a^2 - b^2}$.

265. $w = A(z + \sqrt{[z^2 - (a^2 - b^2)]})$, where A is an arbitrary complex number; $\mu = \frac{a-b}{\sqrt{(a^2+b^2)} - \sqrt{(b^2+k^2)}}$.

266. The whole plane with a cut along the segment $\left[-1, \frac{1}{2}\left(a + \frac{1}{a}\right)\right]$ if $a > 0$; the whole plane with cuts along the rays $\left[-\infty, \frac{1}{2}\left(a + \frac{1}{a}\right)\right]$ and $[-1, +\infty)$ if $a < 0$.

267. $w = \sqrt{\left[\frac{1 + \frac{1}{2}\left(z + \frac{1}{z}\right)}{\frac{5}{4} - \frac{1}{2}\left(z + \frac{1}{z}\right)}\right]}$.

268. $w = \sqrt{\left(\frac{1}{2}\left[\left(a + \frac{1}{a}\right) - \left(z + \frac{1}{z}\right)\right]\right)}$.

269. $w = \frac{\sqrt{\left[\left(z^2 + \frac{1}{z^2}\right) + \left(\alpha^2 + \frac{1}{\alpha^2}\right)\right]}}{z + \frac{1}{z}}$.

270. $w = \sqrt{\left(\frac{1}{2}\left[\left(\alpha^2 + \frac{1}{\alpha^2}\right) + \left(z^2 + \frac{1}{z^2}\right)\right]\right)}$.

271. $\left[1 + \frac{h^2}{4(1-h)}\right]\left(w + \frac{1}{w}\right) = \left(\frac{z}{e^{i\alpha}} + \frac{e^{i\alpha}}{z}\right) - \frac{h^2}{2(1-h)}$.

272. $w + \frac{1}{w} = \frac{4a}{(1+a)^2}\left(z + \frac{1}{z}\right) - 2\frac{(1-a)^2}{(1+a)^2}$; $w'(0) = \frac{(1+a)^2}{4a}$;

the length of the arc corresponding to the cut equals $2\cos^{-1}\frac{6a - 1 - a^2}{(1+a)^2}$; it is equal to π for $a = 3 - \sqrt{8}$.

273. $w + \frac{1}{w} = 2\frac{2\left(z + \frac{1}{z}\right) - \left(a + \frac{1}{a}\right) + \left(b + \frac{1}{b}\right)}{\left(a + \frac{1}{a}\right) + \left(b + \frac{1}{b}\right)}$; $w'(0) = \frac{a + \frac{1}{a} + b + \frac{1}{b}}{4}$;

the lengths of the arcs corresponding to the cuts are equal respectively to:

$$2\cos^{-1}\frac{4-\left(a+\frac{1}{a}\right)+\left(b+\frac{1}{b}\right)}{\left(a+\frac{1}{a}\right)+\left(b+\frac{1}{b}\right)},\quad 2\pi-2\cos^{-1}\frac{-4-\left(a+\frac{1}{a}\right)+\left(b+\frac{1}{b}\right)}{\left(a+\frac{1}{a}\right)+\left(b+\frac{1}{b}\right)}.$$

274. (1) The image of the circle C is a circular arc with ends at the points ± 1, inclined at the point 1 at an angle 2α to the real axis; the exterior of the given circle is mapped onto the whole plane with a cut along the given arc;

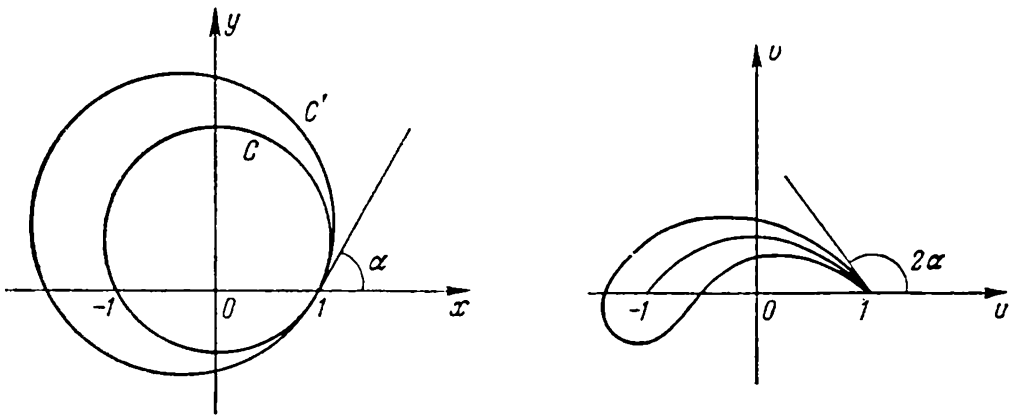

FIG. 66

(2) The image of the circle C is (see Fig. 66) a closed curve (*Zhukovskii profile*) with cusp at $w = 1$, the tangent at this point making an angle 2α with the real axis; the circular arc with ends ± 1 discussed in part (1), is contained in the domain bounded by the Zhukovskii profile; the exterior of the circle C is mapped onto the exterior of the Zhukovskii profile.

275. (1) The image of the circle C is a closed curve consisting of two circular arcs with common ends at the points ± 1, the tangents to these arcs at the point 1 making with the real axis angles respectively equal to $2\alpha-\alpha\delta$ and $2\alpha+(\pi-\alpha)\delta$; the exterior of the circle is mapped onto the exterior of the domain bounded by the given arcs; the image of the circle C' (see Fig. 67) is a closed curve with an angular point at $w = 1$, the tangents at this point being inclined to the real axis at angles respectively equal to $2\alpha-\alpha\delta$ and $2\alpha+(\pi-\alpha)\delta$; the image of the circle C is contained in the domain bounded by the image of the circle C'; the exterior of the circle C' is mapped onto the exterior of the image of this circle.

(2) The interior of the circle C is mapped onto the exterior of the domain bounded by arcs of circles passing through the points $-1,1$, the tangents to these circles at the point 1 forming with the real axis angles respectively equal to: (a) $2\alpha+(\pi-\alpha)\delta$, $2\alpha+(2\pi-\alpha)\delta$, if the function $w(z)$ is defined in the z-plane with a cut along an arc of the circle C lying in the

lower half-plane, (b) $2\alpha-\alpha\delta$, $2\alpha-(\pi+\alpha)\delta$, if the cut defining the function $w(z)$ is made along an arc of the circle C, which lies in the upper half-plane.

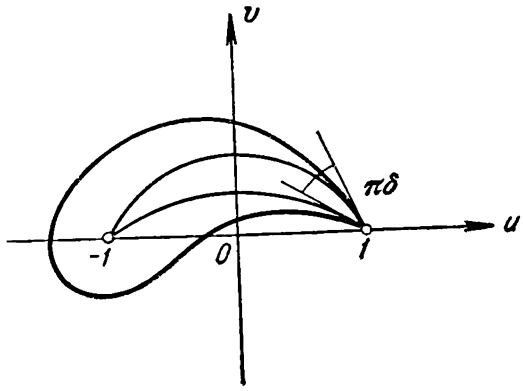

FIG. 67

276. $\dfrac{w-1}{w+1}=\left(\dfrac{ze^{i\gamma}+ie^{i\frac{\beta}{2}}}{ze^{i\gamma}+ie^{-i\frac{\beta}{2}}}\right)^2$, where $\gamma=\alpha$, if $\beta>0$, and $\gamma=\alpha+\pi$ if $\beta<0$.

277. The whole plane with cuts along the rays $y=0$, $x\leqslant -1/2$ and $y=0$, $x\geqslant 1/2$.

278. The half-plane $x>1/2$ with a cut along the segment $y=0$, $1/2\leqslant x\leqslant 1$.

279. The whole plane with cuts along the rays $y=0$, $1\leqslant x<\infty$ and $y=0$, $-\infty<x\leqslant -1$.

280. The angle $-\pi/n<\arg z<\pi/n$ with a cut along the ray $y=0$, $\sqrt[n]{\dfrac{1}{4}}\leqslant x<\infty$.

281. (1) The whole plane with cuts along the rays $|w|\geqslant\dfrac{1}{\sqrt[n]{4}}$, $\arg w=\dfrac{2k\pi}{n}$ $(k=0,\ 1,\ \ldots,\ n-1)$; (2) $w=\dfrac{(1+z^n)^{\frac{2}{n}}}{\sqrt[n]{(4)z}}$.

282. (1) $w=\sqrt{[z^2+\sqrt{(z^4-1)}]}=\dfrac{1}{\sqrt{2}}[\sqrt{z^2+1})+\sqrt{(z^2-1)}]$;

(2) $w=\dfrac{\sqrt{[1+\sqrt{(1-z^4)}]}}{z}=\dfrac{1}{z\sqrt{2}}[\sqrt{(1+z^2)}1+\sqrt{(-z^2)}]$.

283. (1) $w=(\alpha^n+\alpha^{-n})^{-\frac{1}{n}}\sqrt[n]{\{z^n+z^{-n}+\sqrt{[(z^n+z^{-n})^2-(\alpha^n+\alpha^{-n})^2]}\}}$.

Solution. The function $\zeta=\dfrac{1}{2}\left(z^n+\dfrac{1}{z^n}\right)$ maps the sector onto the lower half-

plane, the points α and $\alpha e^{\frac{\pi i}{n}}$ passing into the points $\pm\frac{1}{2}\left(\alpha^n+\frac{1}{\alpha^n}\right)$. Next, it is necessary to compress the half-plane $p=\dfrac{\zeta}{\frac{1}{2}(\alpha^n+\alpha^{-n})}$ and map it onto the unit semicircle $\tau=p+\sqrt{(p^2-1)}$. The function $w=\sqrt[n]{\tau}$ is the one required.

(2) $w=\left(\alpha^{\frac{n}{2}}+\alpha^{-\frac{n}{2}}\right)^{-\frac{2}{n}}\left(z^{\frac{n}{2}}+z^{-\frac{n}{2}}+\sqrt{\left[\left(z^{\frac{n}{2}}+z^{-\frac{n}{2}}\right)^2-\left(\alpha^{\frac{n}{2}}+\alpha^{-\frac{n}{2}}\right)^2\right]}\right)^{\frac{2}{n}}$

284. $w=\sqrt{\left[\dfrac{\sqrt{(z^2+c^2)}+\sqrt{(a^2+c^2)}}{\sqrt{(b^2+c^2)}-\sqrt{(z^2+c^2)}}\right]}$ and

$w=\dfrac{1}{\beta}\{\sqrt{(z^2+c^2)}+\alpha+\sqrt{[\sqrt{(z^2+c^2+\alpha)^2-\beta^2}]}\}$.

where $\alpha=\frac{1}{2}[\sqrt{(a^2+c^2)}-\sqrt{(b^2+c^2)}]$, $\beta=\frac{1}{2}[\sqrt{(a^2+c^2)}+\sqrt{(b^2+c^2)}]$.

285. $w=\sqrt{[\sqrt{(z^2+c^2)}+\sqrt{(a^2+c^2)}]}$.

286. $w=\dfrac{1}{\sqrt{2}}\left(\sqrt{\left[\left(\dfrac{z+i}{z-i}\right)^2+1\right]}+\sqrt{\left[\left(\dfrac{z+i}{z-i}\right)^2-1\right]}\right)$.

287. $w=\sqrt{\left[\dfrac{\sqrt{(z^2-1)}+z-1}{\frac{\sqrt{(\alpha^2+1)}}{\alpha+1}(z+i)-\sqrt{(z^2-1)}}\right]}$.

288. For $b<1$, $w=\sqrt{\left[\dfrac{\sqrt{\left(z_1^2+\cot^2\frac{\alpha}{2}\right)}+\cot\frac{\alpha}{2}}{\sqrt{\left(b_1^2+\cot^2\frac{\alpha}{2}\right)}-\sqrt{\left(z_1^2+\cot^2\frac{\alpha}{2}\right)}}\right]}$;

for $b=1$, $w=\sqrt{\left[\sqrt{\left(z_1^2+\cot^2\frac{\alpha}{2}\right)}+\cot\frac{\alpha}{2}\right]}$;

for $b>1$, $w=\sqrt{\left[\dfrac{\sqrt{\left(z_1^2+\cot^2\frac{\alpha}{2}\right)}+\cot\frac{\alpha}{2}}{\sqrt{\left(z_1^2+\cot^2\frac{\alpha}{2}\right)}+\sqrt{\left(b_1^2+\cot^2\frac{\alpha}{2}\right)}}\right]}$; $z_1=\dfrac{1+z}{1-z}$,

$b_1=\dfrac{1+b}{1-b}$.

289. $w=\dfrac{\sqrt{\left[\left(z^2+\frac{1}{z^2}\right)+\left(b^2+\frac{1}{b^2}\right)\right]}+\sqrt{\left[\left(a^2+\frac{1}{a^2}\right)+\left(b^2+\frac{1}{b^2}\right)\right]}}{\sqrt{\left[\left(a^2+\frac{1}{a^2}\right)-\left(z^2+\frac{1}{z^2}\right)\right]}}$.

290. $w=\dfrac{1}{\sqrt{(2+\sqrt{5})}}(\sqrt{[\sqrt{(z^4+4)}+2]}+\sqrt{[\sqrt{(z^4+4)}-\sqrt{5}]})$.

SOLUTION. The function $\zeta = z^2$ maps the upper half-plane with cuts along the segments $[0, 1+i]$, $[0,-1+i]$ onto the domain of problem 285 which we also map onto the upper half-plane. By the symmetry principle these functions map the domain given in the conditions onto the plane with a cut along some segment. It remains to map the exterior of this segment onto the exterior of the unit circle.

291. $w = \dfrac{i}{\sqrt{2}}[(z+\sqrt{(z^2-1)})^{\frac{\pi}{\alpha}} + (z+\sqrt{(z^2-1)})^{-\frac{\pi}{\alpha}} + 2]^{\frac{1}{2}}$

$$= \frac{i}{\sqrt{2}}[(z+\sqrt{(z^2-1)})^{\frac{\pi}{2\alpha}} + (z+\sqrt{(z^2-1)})^{-\frac{\pi}{2\alpha}}].$$

SOLUTION. By means of the function $\tau = \zeta^{\frac{\pi}{\alpha}}$, where $\zeta = z+\sqrt{(z^2-1)}$ is the inverse function of $z = \dfrac{1}{2}\left(z+\dfrac{1}{z}\right)$, the upper half-plane is mapped onto the domain $|\tau| > 1$, $\operatorname{Im} \tau > 0$, which $\dfrac{1}{2}\left(\tau+\dfrac{1}{\tau}\right)$, maps onto the upper half-plane. Applying the symmetry principle we obtain the mapping of the interior of the right hand branch of the hyperbola onto the whole plane with a cut along the ray $(-\infty, -1]$; this latter domain is easily mapped onto the upper half-plane.

REMARK. The factor $\dfrac{1}{\sqrt{2}}$ does not play any part, since the transformation $w' = kw$ $(k > 0)$ maps the half-plane onto itself.

292. $w = [e^{-i\alpha}(z+\sqrt{(z^2-1)})]^{\frac{\pi}{2\beta}} - [e^{-i\alpha}(z+\sqrt{(z^2-1)})]^{-\frac{\pi}{2\beta}}$, where $\beta = \pi-\alpha$.

293. $w = \left[e^{-i\alpha}\dfrac{z+\sqrt{(z^2-c^2)}}{c}\right]^p$, where $c = \sqrt{(a^2+b^2)}$, $\alpha = \tan^{-1}\dfrac{b}{a}$, $p = \pi/[2\tan^{-1}(a/b)]$.

294. (1) The domain is constructed as follows: the ring $r_1^2 < |w| < r_2^2$ is cut along the segment $r_1^2 \leqslant u \leqslant r_2^2$ of the real axis and to the lower edge of the cut there is attaehed part of the same ring: $r_1^2 < |w| < r_2^2$, $0 \leqslant \arg w < 2\alpha$; if $\alpha = \pi$ then the second ring is complete and its free edge must be attached to the free edge of the first ring (in this case we obtain a two-sheeted ring $r_1^2 < |w| < r_2^2$);

(2) If $\alpha \leqslant 1$, the inequality $|z^2-1| < a$ determines two domains (see problem **34**), each of which is mapped onto the single-sheeted circle $|w-1| < a$; if however $a > 1$, the inequality $|z^2-1| < a$ determines a single domain which is mapped onto the two-sheeted eircle $|w-1| < a$ (in order to construct this two-sheeted cirele it is sufficient to cut two identical specimens of the circle $|w-1| < a$ along any radius and attach the lower edge of the cut on the first specimen to the upper edge of the cut on the second specimen and the upper edge of the cut on the first specimen to the lower edge of the cut on the second specimen).

295. (1) The domain is constructed as follows: to the w-plane cut along the segment $[-1,1]$, there is attached the interior of the ellipse

$\frac{4u^2}{\left(R+\frac{1}{R}\right)^2}+\frac{4v^2}{\left(R-\frac{1}{R}\right)^2}=1$, also cut along the segment $[-1,1]$, the lower edge of the cut plane being attached to the upper edge of the cut ellipse, and the upper edge of the cut plane to the lower edge of the cut ellipse;

(2) The two-sheeted domain $\left|\frac{w-1}{w+1}\right|<R^2$ (the inequality $\left|\frac{w-1}{w+1}\right|<R^2$ defines the interior of a circle if $R<1$; a half-plane for $R=1$ and the exterior of a circle in the case when $R>1$). The cut and the corresponding attachments go along a curve connecting the point $w=1$ to some boundary point of the domain $\left|\frac{w-1}{w+1}\right|<R^2$.

296. (1) and (2). The surface consists of two sheets of the z-plane cut along the segment $[-1,1]$, the lower edge of the cut on the first sheet being attached to the upper edge of the cut on the second, and the upper edge of the cut on the first sheet to the lower edge of the cut on the second.

297. (1) and (2). The surface consists of two sheets cut along rays going respectively from the points $-i$, 0, i to infinity, for example, parallel to the real axis in the positive direction. The lower edges of the cuts of the first sheet are attached to the upper edges of the corresponding cuts of the second sheet, and conversely.

298. The surface consists of three sheets of the z-plane cut along the rays $(-\infty,-1]$ and $[1,\infty)$. Along the ray $(-\infty,-1]$ the attachment proceeds as follows: the upper edge of the cut on the first sheet is attached to the lower edge of the cut on the second sheet, the upper edge of the cut on the second sheet to the lower edge of the cut of the third sheet and the upper edge of the cut on the third sheet to the lower edge of the cut of the first sheet. Along the ray $[1,\infty)$ the lower edge of the cut on the first sheet must be attached to the upper edge of the cut on the second sheet, the lower edge of the cut on the second sheet to the upper edge of the cut on the third sheet and the lower edge of the cut on the third sheet to the upper edge of the cut on the first sheet.

299. (1) The polar net $\varrho=\text{const}$, $\theta=\text{const}$;

(2) The logarithmic spiral $\varrho=c^{\frac{\theta-b}{k}}$ (for $k=0$ the ray $\theta=b$);

(3) The angle $\alpha<\theta<\beta$ (for $\alpha=0$ and $\beta=2\pi$ the plane with a cut along the positive part of the real axis);

(4) The whole plane with a cut along the logarithmic spiral $\varrho=e^{\theta}$;

(5) The sector $\varrho<1$, $0<\theta<\alpha$ (for $\alpha=2\pi$ onto the unit disk with a cut along the radius $v=0$, $0\leqslant u\leqslant 1$);

(6) The domain $\varrho>1$, $0<\theta<\alpha$ (for $\alpha=2\pi$ onto the exterior of the unit circle with a cut along the ray $v=0$. $1\leqslant u<\infty$);

(7) The domain $e^{\alpha}<\varrho<e^{\beta}$, $\gamma<\theta<\delta$ (for $\delta-\gamma=2\pi$ this domain is a concentric ring with a cut along the segment $\theta=\gamma$, $e^{\alpha}\leqslant\varrho\leqslant e^{\beta}$).

300. The angle $0<\arg(z+n)<\pi/n$; the strip $0<y<\pi$.

301. (1) The rectangular Cartesian net $u=C$, $v=C$; (2) A straight line;

(3) The strip $0<v<\alpha$; (4) The half-strip $u<0$, $0<v<\alpha$;

(5) The rectangle $\log r_1<u<\log r_2$, $0<v<2\pi$.

302. (5) $b=a\left|\tanh\frac{\xi_0}{2}\right|$, $l=a\left|\tan\frac{\eta_0}{2}\right|$.

303. (1) The family $x = C$ is transformed into the family of confocal hyperbolas with foci at the points ± 1, $\left(\frac{u^2}{\cos^2 C} - \frac{v^2}{\sin^2 C} = 1\right)$; the family $y = C$ into the family of confocal ellipses with the same foci $\left(\frac{u^2}{\cosh^2 C} + \frac{v^2}{\sinh^2 C} = 1\right)$;

(2) The upper half-plane; (3) The fourth quadrant;

(4) The right half-plane with a cut along the segment $[0, 1]$;

(5) The whole plane with cuts along the real axis along the rays $(-\infty, -1]$ and $[1, \infty)$;

(6) the interior of the ellipse $\frac{u^2}{\cosh^2 h} + \frac{v^2}{\sinh^2 h} = 1$ with cuts along the segments $[-\cosh h, -1]$ and $[1, \cosh h]$.

304. (1) The half-strip $-\frac{\pi}{2} < u < \frac{\pi}{2}$, $v > 0$;

(2) The strip $-\frac{\pi}{2} < u < \frac{\pi}{2}$;

(3) The half-strip $0 < u < \frac{\pi}{2}$, $v > 0$;

(4) The strip $-\frac{\pi}{2} < u < 0$.

305. (1) The family $x = C$ is transformed into the family of confocal ellipses with foci at the points ± 1 $\left(\frac{u^2}{\cosh^2 C} + \frac{v^2}{\sinh^2 C} = 1\right)$; the family $y = C$ into the family of confocal hyperbolas with the same foci $\left(\frac{u^2}{\cos^2 C} - \frac{v^2}{\sin^2 C} = 1\right)$;

(2) The whole plane with cuts along the real axis along the rays $(-\infty, -1]$ and $[1, \infty)$;

(3) The upper half-plane.

306. (1) The strip $-\pi/2 < v < \pi/2$; (2) the half-strip $0 < v < \pi/2$, $u > 0$.

307. (1) The family $x = C$ is transformed into a family of circular arcs with ends at the points $w = \pm 1$, including also the corresponding parts of the imaginary axis; the equation of the pencil of circles: $(u - a)^2 + v^2 = 1 + a^2$ $(a = \cot 2C)$; the family $y = C$ is transformed into the family of circles of Apollonius with respect to the points $w = \pm i$ (including also the real axis); the equation of a family of circles of Apollonius: $u^2 + (v - b)^2 = b^2 - 1$, $|b| > 1$ $(b = \coth 2C)$;

(2) The upper half-plane with a cut along the segment $0 \leqslant v \leqslant 1$ of the imaginary axis;

(3) The whole plane with a cut along the segment $-1 \leqslant v \leqslant 1$ of the imaginary axis;

(4) The semicircle $|w| < 1$, $\operatorname{Re} w > 0$;

(5) The unit circle.

308. (1) The whole plane with two cuts on the real axis: along the rays $(-\infty, -1]$ and $[1, \infty)$;

(2) The right-hand half-plane with a cut along the ray $[1, \infty)$ of the real axis.

309. $w = e^{\frac{\pi(1-i)z}{h}}$.

310. $w = -\cosh\frac{(z-1)\pi}{h}$.

311. $w = e^{\frac{2\pi iz}{z-2}}$.

312. $w = e^{\frac{\pi i(z+2)}{3(z-2)}}$.

313. $w = \cos\frac{\pi(z+2)}{2z}$.

314. $w = e^{\frac{\pi}{\sqrt{(2)}-1}(\sqrt{(z)}-1)}$.

315. (1) $w = -\frac{1+i\sqrt{3}}{2}\tanh\frac{\pi i(z+3i)}{4(z-i)}$;

(2) $w = -\dfrac{e^{\pi i\frac{z+i}{z-i}}+2-i}{e^{\pi i\frac{z+i}{z-i}}+2+i}$;

(3) $w = \dfrac{2e^{\pi i\frac{z+i}{z-i}}}{1+e^{\pi i\frac{z+i}{z-i}}}$.

316. (1) $w = \sqrt{(e^{2\pi iz}+e^{-2\pi h})}$; (2) $w = \sqrt{\left[\frac{e^{2\pi iz}+e^{-2\pi h_1}}{e^{2\pi iz}+e^{-2\pi h_2}}\right]}$.

317. $w = \sqrt{\left[\frac{\cos\pi z-\cos\pi h}{1+\cos\pi z}\right]}$.

318. $w = \sqrt{\left[\frac{\cos\pi z-\cos\pi h_1}{\cos\pi z+\cos\pi h_2}\right]}$.

319. $w = \sqrt{(\cos 2z+\cosh 2h)}$.

320. $w = \sqrt{\left(\frac{\cos 2z+\cosh 2h}{\cos 2z+1}\right)}$.

321. $w = \sqrt{\left(\frac{\cos 2z+\cosh 2h_1}{\cos 2z+\cosh 2h_2}\right)}$.

322. $w = \sqrt{\left(\frac{\sin\frac{\pi}{z}}{1+\sin\frac{\pi}{z}}\right)}$.

323. $w = \sqrt{\left(\frac{1+\cos\frac{4\pi}{z}}{\cos\frac{4\pi}{z}-\cos\frac{4\pi}{a}}\right)}$.

324. $w = \sqrt{\left(\frac{\cos\frac{4\pi}{z}-\cos\frac{4\pi}{b}}{\cos\frac{4\pi}{z}-\cos\frac{4\pi}{a}}\right)}$.

325. $w = \sqrt{\left(\frac{\cos\frac{2\pi}{z}-\cos\frac{2\pi}{b}}{\cos\frac{2\pi}{z}-\cos\frac{2\pi}{a}}\right)}$.

326. $w = \sqrt{\left(\frac{e^{-\frac{2\pi}{\beta}}-e^{\frac{2\pi i}{z}}}{e^{2\pi/a}-e^{2\pi i/z}}\right)}$.

327. $w = \sqrt{\left(\frac{\cosh\frac{\pi}{h}-\cos\frac{\pi}{z}}{1-\cos\frac{\pi}{z}}\right)}$.

328. $w = i\cosh\frac{\pi\sqrt{z}}{2\alpha}$, $w = \tanh^2\frac{\pi\sqrt{z}}{4\alpha}$

(see the remark to the answer to problem **291**).

329. $w = \sin^{-1}\frac{\sin z}{\cosh\alpha}$. The function $\sin z$ maps the half-strip $y>0$, $-\pi/2 < x < \pi/2$ onto the upper half-plane: the points $\pm\pi/2+ai$ then become the

points $\pm \cosh a$. Hence it is easily obtained that the function $w = \sin^{-1} \dfrac{\sin z}{\cosh a}$ will map the given half-strip onto itself so that the rays $x = \pm \pi/2$, $a \leqslant y < \infty$ will correspond to the rays $u = \pm \pi/2$, $0 \leqslant v < \infty$. On applying the symmetry principle an infinite number of times we see that the function found is the one required.

330. $w = \dfrac{b \sin^{-1} \dfrac{i \sinh z}{\cosh a}}{\sin^{-1} (1/\cosh a)}$.

331. $w = \dfrac{\sin^{-1} \dfrac{1}{\cosh a} + \sqrt{\left[\left(\sin^{-1} \dfrac{1}{\cosh a}\right)^2 - \left(\sin^{-1} \dfrac{\sin z}{\cosh a}\right)^2\right]}}{\sin^{-1} \dfrac{\sin z}{\cosh a}}$.

332. $w = \sqrt{\left[\dfrac{\sin^{-1} \dfrac{\sin z}{\cosh a} - \sin^{-1} \dfrac{\sin p}{\cosh a}}{\sin^{-1} \dfrac{\sin q}{\cosh a} - \sin^{-1} \dfrac{\sin z}{\cosh a}}\right]}$.

333. $w = \sin^{-1} e^{2iz}$. Solution. The function $\zeta = e^{2iz}$ maps the strip $0 < x < \pi/2$ onto the upper half-plane and the function $w = \sin^{-1} \zeta$ maps the upper half-plane onto the half-strip $-\pi/2 < u < \pi/2$, $v > 0$. The function $w = \sin^{-1} e^{2iz}$ which maps the strip onto the half-strip transforms the rays $x = 0$, $-\infty < y < 0$; $x = \pi/2$, $-\infty < y < 0$ into the rays $x = \pi/2$, $0 < y < \infty$; $x = \pi/2$, $0 < y < \infty$, respectively. Applying the symmetry principle an infinite number of times we see that this funetion effects the required mapping.

334. (1) If $b < 2\pi$ the curvilinear rectangle: $1 < \varrho < e^a$, $0 < \theta < b$; if $b = 2\pi$ the ring $1 < \varrho < e^a$ with a cut along the segment $[1, e^a]$; if $b = 2k\pi$ ($k = 2, 3, \ldots$) a many-sheeted domain consisting of k rings $1 < \varrho < e^a$, cut along the segment $[1, e^a]$ and attached in such a way that the lower edge of the cut of the first ring is attached to the upper edge of the cut of the second ring, the lower edge of the cut of the second ring to the upper edge of the cut of the third ring and so on; if $b = 2k\pi + \beta$ ($k = 2, 3, \ldots$, $0 < \beta < 2\pi$), then to the free lower edge of the last ring of the surface being constructed it is necessary to attach along the segment $[1, e^a]$ the curvilinear rectangle $1 < \varrho < e^a$, $0 < \theta < \beta$;

(2) The domain with an infinite number of sheets consisting of rings $1 < \varrho < e^a$, cut along the segment $[1, e^a]$ and attached by the method indicated above;

(3) The infinitely many-sheeted domain consisting of the rings $1 < \varrho < e^a$ cut along the segment $[1, e^a]$, enumerated by means of the integers $(\ldots, -2, -1, 0, 1, 2, \ldots)$ and attached so that the lower edge of the cut of each ring is attached to the upper edge of the cut of the ring the number of whieh is one unit higher. This domain is the part of the Riemann surface of the function Log w which lies above the ring $1 < \varrho < e^a$.

335. (1) The two-sheeted domain obtained by joining together two right half-planes, each of which is cut along the ray $v = 0$, $1 \leqslant u < \infty$; the edges of the cuts are attached criss cross, that is, so that the lower edge of the cut of the first sheet is attaehed to the upper edge of the cut of the second sheet, and conversely;

(2) The two-sheeted domain consisting of two planes with cuts along the real axis along tho rays $-\infty < u \leqslant -1$ and $1 \leqslant u < \infty$ and attached criss cross along the cuts $-\infty < u \leqslant -1$. The edges of the cuts $1 \leqslant u < \infty$ remain free.

336. A two-sheeted domain consisting of two planes cut along the imaginary axis along the segment $-1 \leqslant v \leqslant 1$ and attached so that the left-hand edge of the cut of the first sheet is attached to the right-hand edge of the cut of the second sheet. The remaining edges are free.

337. The Riemann surface is infinitely many-sheeted and has two logarithmic branch points above the points $w = 0$ and $w = \infty$. Domains of single-valuedness in the z-plane corresponding to sheets of the w-plane with cuts along the positive real axis are bounded by the circles $2k\pi(x^2+y^2)+y = 0$ ($k = 0, \pm 1, \pm 2, \ldots$).

338. $r_1 = \infty$, $r_2 = r_3 = 1$; the image of the disk $|z| < r_1$ is the whole plane with the point $w = -1$ deleted; the image of the disk $|z| < r_2$ (and of the disc $|z| < r_3$) is the half-plane $\operatorname{Re} w > -\frac{1}{2}$.

339. (1) $r_1 = \frac{1}{2}$; (2) $r_1 = \frac{1}{2|a|}$; (3) $r_1 = 1$.

343. (1) $r_2 = 1/4$; (2) $r_2 = \frac{1}{4|a|}$; (3) $r_2 = 2 - \sqrt{3}$.

344. $r_1 = \pi$, $r_2 = 1$.

345. (1) $r_3 = 1/2$; (2) $r_3 = \frac{1}{2|a|}$; (3) $r_3 = 1$.

CHAPTER III

347. (1) For example, the whole plane with a segment removed;

(2) For example, the domain bounded by two circles touching internally.

354. SOLUTION. The Jordan curve γ can be considered as the topological image of the segment Δ (or, if the curve is closed of the circle Δ). Lot us assume that γ contains the circle $K : |z - z_0| < \varrho$. On Δ the circle K corresponds to some continuum, consequently, the segment $[a, b]$. The arc $\gamma_{ab} \subset K$ joining the images of the points a and b on γ, also corresponds on Δ to a continuum, moreover it belongs to $[a, b]$ and contains the points a, b, consequently it coincides with $[a, b]$. But this is impossible. However, continuous curves exist which contain interior points†.

358. SOLUTION. (1) We divide up γ into sufficiently small parts, draw chords, map them linearly onto the corresponding parts of the segment $0 \leqslant t \leqslant 1$ and obtain the polygonal path π; then map the segments connecting points on γ and π which correspond to identical values of t, continuously onto the segment $0 \leqslant \lambda \leqslant 1$;

(2) and (3). We first replace γ (in the circle or in the ring) by a homotopic polygonal path.

362. SOLUTION. Let O be the centre of Q, γ an arbitrary closed path in Q not passing threugh O, β an auxiliary circle with centre at the point O such that the path γ lies in the ring between β and Q, and $\delta = n_{\beta'}(O')$, where β', O' are the images of β, O. In the circle Q the path γ is homotopic to an n_γ (O) times traversed circle β (see, **358**, (3)) consequently, γ' the image of γ is homotopic in Q' to an $n_\gamma(O)$ times traversed path β', hence $n_{\gamma'}(O') = n_\gamma(O)\,\delta$. In particular, choosing as γ the original of some cirele γ' around O', we have $n_{\gamma'}(O') = 1$, whence we conclude that $\delta = \pm 1$. Now let a be an arbitrary point of Q. As above we conclude that $n_{\gamma'}(a') = n_\gamma(a)\,\delta_a$, where δ_a has a meaning similar to δ for O. We show that $\delta_a = \delta$. For this we draw in Q the circle β, enelosing O and a. Then: $n_\beta(a) = n_\beta(O) = 1$, $n_{\beta'}(a') = n_{\beta'}(O)$, $n_{\beta'}(a') = \delta_a$, $n_{\beta'}(O') = \delta$, whence it follows that $\delta_a = \delta$. In the case of a continuously differentiable mapping with a Jacobian different from zero the infinitesimal behaviour is equivalent to an affine transformation (see problem **375**).

363. SOLUTION. Divide up the domain G by Jordan arcs into the domains $G^1_{i_1}$ $(i_1 = 1, 2, \ldots, n_1)$ with diameters less than 1. If $f(z) \neq a$, on these arcs then denoting by γ_{i_1} the boundary of $G^1_{i_1}$, we have

$$\int_\gamma \mathrm{d}\arg[f(z)-a] = \sum_{i_1=1}^{n_1} \int_{\gamma_{i_1}} \mathrm{d}\,[\arg f(z)-a] \neq 0,$$

hence a domain $G^1_{s_1}$ (s_1 is the value ef i_1) has been found, such that

$$\int_{\gamma_{s_1}} \mathrm{d}\,[\arg f(z)-a] \neq 0.$$

† See, for example, A.S. PARKHOMENKO, *What is a curve?* (*Chto takoe liniya*), Chapter I, § 2, Gostekhizdat, 1954.

Divide up $G^1_{s_1}$ by Jordan arcs into domains $G^2_{s_1 i_2}$ ($i_2 = 1, 2, \ldots, n_2$) with diameters less than 1/2, then, repeating the process, construct the domains $G^3_{s_1 s_2 i_3}$ with diameters less than 1/3, and so on. Then either on one of the arcs a point will be found where $f(z) = a$, or we obtain a sequence of domains $G^1_{s_1}$, $G^2_{s_1 s_2}$, $G^3_{s_1 s_2 s_3}, \ldots$, enclosed one within the other which shrink to some point $z_0 \in \overline{G}$, consequently the images of these domains in the w-plane shrink to the point $f(z_0)$ and as all the time $\int \mathrm{d} \arg [f(z)-a] \neq 0$ along the boundary of these domains, it follows that $f(z_0) = a$. Since by supposition $f(z) \neq a$ on γ, it follows that z_0 lies within G.

375. $a_1 = b_2$, $b_1 = -a_2$ or $a_1 = -b_2$, $b_1 = a_2$.

384. It is possible to construct the required quasi-conformal transformation with the characteristic $p \leqslant \dfrac{R+|a|}{R-|a|}$.

385. $u = x/(\cos\alpha)$, $v = y - x \tan\alpha$, $p = (1+\sin\alpha)/\cos\alpha$.

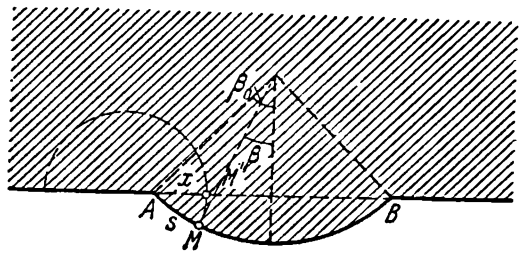

FIG. 68

386. SOLUTION. By means of the function $\zeta = \log [(z-a)/(z-b)]$ (the values $z = a$ and $z = b$ correspond to the points A and B) we conformally map the given domain onto a strip of width $\pi+\beta_0$. We compress this strip into a strip of width π (a quasi-conformal mapping with characteristic $p = 1 + \dfrac{\beta_0}{\pi}$) and, using the inverse function $z(\zeta)$, we map the latter strip onto the half-plane. Then the arc $\overset{\smile}{AM}$ (Fig. 68) of length s will occupy the position of the segment AM' of length x; the point M moves along the circle of Apollonius with respect to the points A and B (shown dotted), we have $\dfrac{\mathrm{d}x}{\mathrm{d}s} = \dfrac{\cos^2\frac{\beta_0}{2}}{\cos^2\frac{\beta}{2}} \geqslant \cos^2\frac{\beta_0}{2}$ (prove this!). In the resulting half-plane we widen the vertical half-strip on the segment AB as base to the length of the arc $\overset{\smile}{AB}$, and moreover in such a way as in the result to preserve lengths everywhere on the arc $\overset{\smile}{AB}$. The resulting quasi-conformal mapping has the characteristic $p \leqslant \left(1+\dfrac{\beta_0}{\pi}\right)\sec^2\dfrac{\beta_0}{2}$.

388. $A = \frac{1}{4}[(a-d)+i(c+b)]$, $B = \frac{1}{4}[(a+d)+i(c-b)]$, $F = \frac{1}{2}(f+ig)$.

389. $b = B\bar{a}$. In particular, it is possiblo to take $w = \omega + B\overline{\omega}$, and also $\omega_1 = w - B\overline{w}$.

391. The highest terms of the equation of the problem determine a quasi-conformal mapping with two pairs of characteristics (P, Q), (P_1, Q_1), depending on $q_1(z)$ and $q_2(z)$. The transformation $\zeta(z)$ is a quasi-conformal mapping with one pair of characteristics $p\theta$ and $q_1^* = -\frac{p-1}{p+1}e^{2i\theta}$ (see problem **378**). The coefficient $q_2^* = \frac{p_1-1}{p_1+1}e^{2i\theta_1}$ (see problem **379**).

CHAPTER IV

397. (1) $I_1 = 2+i$, $I_2 = 1+\frac{i}{2}$; (2) $I_1 = \frac{i\pi}{2}$, $I_2 = -\frac{\pi}{2}$; (3) $I_1 = i\pi R^2$, $I_2 = -\pi R^2$.

398. (1) $\sqrt{5}\left(1-\frac{i}{2}\right)$; (2) 2; (3) $2i$; (4) 0.

399. πi.

400. 4/3.

401. (1) $\frac{R^{n+1}}{n+1}[(-1)^{n+1}-1]$, if $n \neq -1$; πi, if $n = -1$;
(2) and (3). 0, if $n \neq -1$; $2\pi i$, if $n = -1$.

402. (1) $-2(1-i)$; (2) $2(1-i)$; (3) $-2(1+i)$; (4) -4; (5) $4i$.

403. (1) $2\pi i$; (2) -2π; (3) $2\pi Ri$; (4) $2\pi Ri$.

404. (1) $\frac{2\pi i}{n+1}$, if $n \neq -1$; $-2\pi^2$, if $n = -1$; (2) $(-1)^{n+1}\frac{2\pi i}{n+1}$ if $n \neq -1$; $-2\pi^2$ if $n = -1$.

405. $\frac{e^{2\alpha\pi i}-1}{1+\alpha}$, if $\alpha \neq -1$; $2\pi i$, if $\alpha = -1$.

407. (1) $|\alpha| \leqslant \pi/2$; (2) $\sin p\alpha \geqslant 0$.

421. (1) $\pi/3$; (2) $-\pi/3$; (3) 0.

422. If the contour C contains the point 0 as an interior point and does not contain 1 and -1 then $I = -2\pi i$; if it contains only one of the points -1 or 1 and does not contain the point 0 then $I = \pi i$. It is clear from this that the integral may assume five different values ($-2\pi i$; $-\pi i$; 0; πi; $2\pi i$).

423. $2^n - 1$, if $n > 1$; 2, if $n = 1$.

424. $\pi i/2$.

425. $(\sin a)/a$.

426. $e^a\left(1+\frac{a}{2}\right)$.

427. (1) 1; (2) $-\frac{1}{2}e$; (3) $1-\frac{1}{2}e$.

429. (1) 2/3; (2) $1 - 2i/3$.

445. Convergent.

446. Convergent.

447. Divergent.

448. Non-absolutely convergent.

449. Non-absolutely convergent for $\phi \neq 2k\pi$ ($k = 0, \pm 1, \pm 2, \ldots$), divergent for $\phi = 2k\pi$.

450. Converges absolutely.

451. Diverges.

452. Absolutely convergont.

453. Divergent.

454. Absolutely convergent.

455. $R = 1$.

456. ∞.

457. 0.

458. 2.

459. e.

460. 1.

461. 1.

462. 1.

463. 1/4.

464. 1/e.

465. 1, if $|a| \leqslant 1$; $1/|a|$, if $|a| > 1$.

466. 1.

467. (1) R; (2) $R/2$; (3) ∞; (4) 0; (5) R^k; (6) R, if $|z_0| \leqslant 1$, and $R/|z_0|$, if $|z_0| > 1$.

468. (1) $R \geqslant \min\ (r_1,\ r_2)$; (2) $R \geqslant r_1 r_2$; (3) $R \leqslant r_1/r_2$.

469. (1) $z/(1-z)^2$; (2) $-\log\,(1-z)$; (3) $\frac{1}{2}\log\,[(1+z)/(1-z)]$; (4) $\log\,(1+z)$.

470. Divergent at every point.

471. Convergent (non-absolutely) at every point except $z=1$.

472. Absolutely convergent.

473. Convergent (non-absolutely) at every point except $z=-1$.

474. Converges (non-absolutely) at every point except the points $z = e^{\frac{2k\pi i}{p}}$ $(k = 0, 1, \ldots, p-1)$.

475. Converges (non-absolutely) at every point except the points $z = (1 \pm i\sqrt{3})/2$ and $z = -1$.

476. Converges absolutely.

477. For example, $c_n = (-1)^n$.

479. SOLUTION. Let us first investigate the convergence at the point $z = 1$. The terms of the series $\sum_{n=1}^{\infty} \frac{(-1)^{[\sqrt{n}]}}{n}$ with denominators $k^2,\ k^2+1,\ \ldots,$ $(k+1)^2-1$ have the sign $(-1)^k$; let us denote the sum of these terms by $(-1)^k\sigma_k$ and prove that $\sigma_k \to 0$ monotonically. We have

$$0 < \sigma_k < \frac{(k+1)^2-k^2}{k^2} = \frac{2k+1}{k^2} \to 0 \text{ as } k \to \infty.$$

On the other hand

$$\sigma_k - \sigma_{k+1} = \left[\frac{1}{k^2} - \frac{1}{(k+1)^2}\right] + \left[\frac{1}{(k^2+1)} - \frac{1}{(k+1)^2+1}\right] + \ldots$$

$$\ldots + \left[\frac{1}{(k+1)^2-1} - \frac{1}{(k+2)^2-3}\right] - \frac{1}{(k+2)^2-2} - \frac{1}{(k+2)^2-1}$$

$$> (2k+1)\left[\frac{1}{(k+1)^2-1} - \frac{1}{(k+2)^2-3}\right] - \frac{1}{(k+2)^2-2} - \frac{1}{(k+2)^2-1} > 0.$$

This proves that the given series converges for $z = 1$. If $|z| = 1$, but $z \neq 1$, then use must be made of the convergenee test of problem **440**, putting $a_n = (-1)^{[\sqrt{n}]}z^n$, $b_n = \frac{1}{n}$. We have

$$s_n = -z-z^2-z^3+z^4+ \ldots +(-1)^{[\sqrt{n}]}z^n = -z\frac{1-z^3}{1-z} + z^4\frac{1-z^5}{1-z} - z^9\frac{1-z^7}{1-z} + \ldots$$

$$\ldots \pm z^{p^2}\frac{1-z^{2p+1}}{1-z} \pm (z^{(p+1)^2} + \ldots + z^n)$$

where $p = [\sqrt{n}] - 1$.

Hence $|s_n| < \frac{2p}{|1-z|} + 2p + 3$, consequently for every z there exists a k such that $|s_n| < kp < k\sqrt{n}$ and this eempletes the proof.

482. $\sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$, $R = \infty$.

483. $\sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$, $R = \infty$.

484. $\sum_{n=0}^{\infty}(-1)^{n+1}\frac{2^{2n-1}z^{2n}}{(2n)!}$, $R=\infty$. **485.** $1/2+\sum_{n=0}^{\infty}\frac{2^{2n-1}z^{2n}}{(2n)!}$, $R=\infty$.

486. $a^{\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n}\left(\frac{z}{a}\right)^n$ where $\binom{\alpha}{0}=1$, $\binom{\alpha}{n}=\frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!}$ $(n=1,2,\ldots)$; $R=|a|$.

487. $\frac{1+i}{\sqrt{2}}\left[1+\frac{1}{2}\frac{z}{i}+\sum_{n=2}^{\infty}(-1)^{n-1}\frac{1.3\ldots(2n-3)}{2.4\ldots 2n}\left(\frac{z}{i}\right)^n\right]$, $R=1$.

488. $\sum_{n=0}^{\infty}(-1)^n\frac{a^nz^n}{b^{n+1}}$, $R=\left|\frac{b}{a}\right|$.

489. $\frac{i}{6}\sum_{n=1}^{\infty}\frac{(2-3i)^n-(2+3i)^n}{13^n}z^n=\sum_{n=1}^{\infty}c_nz^n$,

where $c_n=\sum_{m=0}^{\left[\frac{n-1}{2}\right]}(-1)^m\binom{n}{2m+1}2^{n-2m-1}3^{2m}$, $R=\sqrt{13}$.

490. $\sum_{n=2}^{\infty}(-1)^n(n-1)z^n$, $R=1$.

491. $2\sum_{n=0}^{\infty}\frac{z^{2n+1}}{2n+1}$, $R=1$.

492. $\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{2n+1}$, $R=1$.

493. $\sum_{n=0}^{\infty}(-1)^n\frac{1.3.5\ldots(2n-1)}{2^n.n!\,(2n+1)}z^{2n+1}$, $R=1$.

494. $\log 2-\sum_{n=1}^{\infty}\left(1+\frac{1}{2^n}\right)\frac{z^n}{n}$, $R=1$.

495. $\sum_{n=0}^{\infty}\frac{z^{2n+1}}{n!(2n+1)}$, $R=\infty$.

496. $\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{(2n+1)!\,(2n+1)}$, $R=\infty$.

497. $\frac{1}{3}+2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(z-1)^n}{3^{n+1}}$, $R=3$.

498. $\frac{1}{4}\left[1+\frac{z-1}{2^2}+\frac{(z-1)^2}{2^2}+\frac{(z-1)^3}{2^4}+\frac{(z-1)^4}{2^4}+\ldots\right.$

$\left.\ldots+\frac{(z-1)^{2k-1}}{2^{2k}}+\frac{(z-1)^{2k}}{2^{2k}}+\ldots\right]\left(=\frac{1}{4}\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^{n+\frac{1}{2}[(-1)^{n+1}+1]}}\right)$, $R=2$.

499. $\frac{1}{4}+\sum_{n=1}^{\infty}(-1)^n\frac{(n-3)(z-1)^n}{2^{n+2}}$, $R=2$.

500. $\frac{-1+i\sqrt{3}}{2}\sum_{n=0}^{\infty}\binom{1/3}{n}(z-1)^n$, $R=1$.

501. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(z-1)^n}{n}$, $R=1$.

502. $\sum_{n=0}^{\infty}\frac{\sin\left(1+\frac{n\pi}{2}\right)}{n!}(z-1)^{2n}$, $R=\infty$.

503. $1+z^2+\frac{z^4}{3}+\ldots$

504. $1-\frac{z^2}{4}-\frac{z^4}{96}+\ldots$

505. $1+z^2-\frac{1}{2}z^3+\frac{5}{6}z^4-\frac{3}{4}z^5+\ldots$

506. $1+z+z^2+\frac{5}{6}z^3-\frac{5}{8}z^4+\frac{13}{30}z^5+\ldots$

507. $z+\frac{z^2}{2!}+\frac{2z^3}{3!}+\frac{9z^5}{5!}+\ldots$

508. $\log 2+\frac{z}{2}+\frac{z^2}{8}-\frac{z^4}{192}+\ldots=\log 2+\sum_{n=0}^{\infty}\frac{c_n}{n+1}z^{n+1}$;

$c_0=\frac{1}{2}$, $c_0+nc_1+n(n-1)c_2+\ldots+n!c_{n-1}+2n!c_n=1$, $n\geqslant 1$.

509. $2\left[\frac{z^2}{2}+\left(1+\frac{1}{2}\right)\frac{z^3}{3}+\left(1+\frac{1}{2}+\frac{1}{3}\right)\frac{z^4}{4}+\ldots+\left(1+\frac{1}{2}+\frac{1}{3}+\ldots\right.\right.$

$\left.\left.\ldots+\frac{1}{n-1}\right)\frac{z^n}{n}+\ldots\right]$.

510. $-4\pi^2+2\left[-2\pi iz+(1-2\pi i)\frac{z^2}{2}+\left(1+\frac{1}{2}-2\pi i\right)\frac{z^3}{3}+\ldots\right.$

$$\left.\ldots+\left(1+\frac{1}{2}+\ldots+\frac{1}{n-1}-2\pi i\right)\frac{z^n}{n}+\ldots\right].$$

511. $z^2-\left(1+\frac{1}{3}\right)\frac{z^4}{2}+\left(1+\frac{1}{3}+\frac{1}{5}\right)\frac{z^6}{3}-\ldots+(-1)^{n+1}\left(1+\frac{1}{3}+\frac{1}{5}+\ldots\right.$

$\left.+\frac{1}{2n-1}\right)\frac{z^{2n}}{n}+\ldots\Big]$.

512. $2\left[\left(1+\frac{1}{2}\right)\frac{z^3}{3}-\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)\frac{z^5}{5}+\ldots+(-1)^{n+1}\left(1+\frac{1}{2}+\frac{1}{3}+\right.\right.$

$\left.\left.\frac{1}{4}+\ldots+\frac{1}{2n}\right)\frac{z^{2n+1}}{2n+1}+\ldots\right]$.

513. $1+\sum_{n=1}^{\infty}\left[\sum_{k=1}^{\infty}\frac{1}{k!}\binom{n-1}{k-1}\right]z^n$.

517. $1+\sum_{k=1}^{\infty}(-1)^k\frac{2^{2k}B_{2k}z^{2k}}{(2k)!}$, $R=\pi$.

518. (1) $\sum_{k=1}^{\infty}(-1)^k\frac{2^{2k}B_{2k}}{2k(2k)!}z^{2k}$, $R=\pi$; (2) $\sum_{k=1}^{\infty}(-1)^{k-1}\frac{2^{2k}(2^{2k}-1)B_{2k}}{(2k)!}z^{2k-1}$,

$R=\frac{\pi}{2}$; (3) $\sum_{k=1}^{\infty}(-1)^k\frac{2^{2k}(2^{2k}-1)B_{2k}}{2k(2k)!}z^{2k}$; $R=\frac{\pi}{2}$;

(4) $1+\sum_{k=1}^{\infty}(-1)^{k-1}\frac{(2^{2k}-2)B_{2k}}{(2k)!}z^{2k}$, $R=\pi$.

519. $c_n=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}-1\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$, $n\geqslant 0$, $R=\frac{\sqrt{5}-1}{2}$.

520. $c_0=\frac{A}{\alpha}$, $c_1=\frac{\alpha B-\beta A}{\alpha^2}$, $c_2=\frac{\alpha^2C-\alpha\beta B+(\beta^2-\alpha\gamma)A}{\alpha^3}$;

$\alpha c_n+\beta c_{n-1}+\gamma c_{n-2}+\delta c_{n-3}=0$ $(n=3, 4, \ldots)$.

526. $\frac{e^{-\frac{zt}{1-t}}}{1-t}$; $L_{n+1}(z)-(2n+1-z)L_n(z)+n^2L_{n-1}(z)=0$ $(n\geqslant 1)$†.

527. $z+z^3+\frac{z^5}{4.5}+\frac{z^9}{4.5.8.9}+\ldots+\frac{z^{4n+1}}{4.5.8.9\ldots 4n(4n+1)}+\ldots$; $|z|<\infty$.

† See the literature mentioned on page 82.

528. $z-\dfrac{n(n+1)-1.2}{3!}z^3+\dfrac{[n(n+1)-1.2][n(n+1)-3.4]}{5!}z^5+\ldots$

$\ldots+\dfrac{[n(n+1)-1.2][n(n+1)-3.4]\ldots[n(n+1)-(2k-1)\,2k]}{(2k+1)!}z^{2k+1}+\ldots;\ |z|<1.$

529. $z/(1-z^2)$.

530. $1-\dfrac{m^2}{2!}z^2+\dfrac{m^2(m^2-2^2)}{4!}z^4-\ldots+(-1)^m\dfrac{m^2(m^2-2^2)\ldots[m^2-(2n-2)^2]}{(2n)!}\times$

$\times z^{2n}+\ldots$

531. $w=F(a,b,c,z)=1+\dfrac{ab}{1.c}z+\dfrac{a(a+1)b(b+1)}{2!\,c(c+1)}z^2+\ldots$

$\ldots+\dfrac{a(a+1)\ldots(a+n-1)b(b+1)\ldots(b+n-1)}{n!\,c(c+1)\ldots(c+n-1)}z^n;\ |z|<1.$

533. SOLUTION. After differentiating the hypergeometric equation we obtain the equation satisfied by the function $\zeta=\dfrac{\mathrm{d}}{\mathrm{d}z}F(a,b,c,z)$:

$$z(1-z)\frac{\mathrm{d}^2\zeta}{\mathrm{d}z^2}+[(c+1)-(a+b+3)z]\frac{\mathrm{d}\zeta}{\mathrm{d}z}-(a+1)(b+1)\zeta=0. \qquad (1)$$

Since the function $\dfrac{\mathrm{d}}{\mathrm{d}z}F(a,b,c,z)$, being the derivative of the function $F(a,b,c,z)$ which is analytic at the point $z=0$ is also analytic at the point $z=0$, and every solution of equation (1) analytic at the point $z=0$ must be of the form $kF(a+1,\ b+1,\ c+1,\ z)$ (see problem **531**), where k is a constant, it follows that $\dfrac{\mathrm{d}}{\mathrm{d}z}F(a,b,c,z)=kF(a+1,\ b+1,\ c+1,\ z)$. Putting $z=0$, we find that $k=ab/c$.

538. (2) $2\pi^2M$.

545. (1) 4; (2) 15; (3) 3.

546. (1) A zero of order $k+l$; (2) A zero the order of which is not lower than min (k,l); (3) A zero of order $k-l$, if $k>l$; a regular point not being a zero if $k=l$, and a singularity if $k<l$.

547. The points $z=\pm 3i$ are zeros of the first order.

548. The points $z=\pm 3i$ are zeros of the first order; the point at infinity is a zero of the second order.

549. $z=0$ is a zero of the seeond order; $z=k\pi$ $(k=\pm 1,\ \pm 2,\ \ldots)$ are zeros of the first order.

550. $z=\pm 2$ are zeros of the third order; $z=2k\pi i$ $(k=0,\ \pm 1,\ \pm 2,\ \ldots)$ are zcros of the first order.

551. $z=2k\pi$ $(k=0,\ \pm 1,\ \pm 2,\ \ldots)$ are zeros of the second order.

552. $z=\pm\pi$ are zeros of the third order; all the remaining points of the form $z=k\pi$ $(k=0,\ \pm 2,\ \pm 3,\ \ldots)$ are zeros of the first order.

553. $z=\pi/4+k\pi$ $(k=0,\ \pm 1,\ \pm 2,\ \ldots)$ are zeros of the first order.

554. It has no zeros.

555. $z=k\pi$ $(k=0,\ \pm 1,\ \pm 2,\ \ldots)$ are zeros of the third order.

556. $z=0$ is a zero of the second order; $z=k\pi$ $(k=\pm 1,\ \pm 2,\ \ldots)$ are zcros of the third order.

557. $z = 0$ is a zero of the third order; $z = \sqrt[3]{(k\pi)}$ and $z = \frac{1}{2}\sqrt[3]{(k\pi)}\,(1 \pm i\sqrt{3})$ $(k = \pm 1, \pm 2, \ldots)$ are zeros of the first order.

558. $z = (2k+1)\frac{\pi}{2}$ $(k = 0, \pm 1, \pm 2, \ldots)$ are zeros of the third order.

559. $z = \sqrt[3]{\left[(2k+1)\frac{\pi}{2}\right]}$ and $z = \frac{1}{2}\sqrt[3]{\left[(2k+1)\frac{\pi}{2}\right]}$ $(1 \pm i\sqrt{3})$ $(k = 0, \pm 1, \pm 2, \ldots)$ are zeros of the third order.

560. $z = 4$ is a zero of the third order for one of the branches.

561. Here two functions are given: one of them has zeros of the second order at the points $z = 2k\pi \pm \pi/6$, the other has zeros of the second order at the points $z = (2k+1)\pi \pm \pi/6$ $(k = 0, \pm 1, \pm 2, \ldots)$

563. (1), (2) and (3). It does not exist; (4) It exists $\left(f(z) = \frac{1}{z+1}\right)$.

564. (1) It exists $(f(z) = z^2)$; (2) it does not exist.

565. There is no contradiction, as the point $z = 1$ does not belong to the domain of analyticity of the function.

566. The limit point can only be the point at infinity.

567. SOLUTION. From the expansion $f(z) = \sum_{n=0}^{\infty} c_n(z - z_0)^n$ it follows that

$$u(x, y) = \frac{c_0+\bar{c}_0}{2} + \sum_{n=1}^{\infty} \{(c_n[(x-x_0)+i(y-y_0)]^n + \bar{c}_n[(x-x_0)-i(y-y_0)]^n\}.$$

Putting here $x = x_0 + \frac{\zeta - z_0}{2}$, $y = y_0 + \frac{\zeta - z_0}{2i}$, where ζ is sufficiently close to z, we obtain (establish this!)

$$u\left(x_0 + \frac{\zeta-z_0}{2},\ y_0 + \frac{\zeta-z_0}{2i}\right) = \frac{1}{2}\,[\bar{c}_0 + f(\zeta)]$$

and, after replacing ζ by z, we arrive at the equality already proved.

569. z^3+2+Ci.

570. $ze^z - i/z+Ci$.

571. $(1+i)z - 3i+C$.

572. $\sin z - \cosh z+C$ (C is an arbitrary real constant).

CHAPTER V

573. $-\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$ for $|z|<2$; $\sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}}$ for $|z|>2$.

574. $\frac{(-1)^k}{a^k}\sum_{n=0}^{\infty}\binom{n+k-1}{k-1}\left(\frac{z}{a}\right)^n$ for $|z|<|a|$; $\frac{1}{z^k}\sum_{n=0}^{\infty}\binom{n+k-1}{k-1}\left(\frac{a}{z}\right)^n$ for $|z|>|a|$.

575. $\frac{1}{z}+\sum_{n=0}^{\infty}z^n$ for $|z|<1$; $-\frac{1}{z-1}+\sum_{n=0}^{\infty}(-1)^n(z-1)^n$ for $0<|z-1|<1$; $-\sum_{n=2}^{\infty}\frac{1}{z^n}$ for $|z|>1$.

576. $\frac{1}{b-a}\sum_{n=0}^{\infty}\frac{b^{n+1}-a^{n+1}}{a^{n+1}b^{n+1}}z^n$ for $|z|<|a|$;

$\frac{1}{a-b}\left[\frac{1}{z-a}+\sum_{n=0}^{\infty}\frac{(z-a)^n}{(b-a)^{n+1}}\right]$ for $0<|z-a|<|b-a|$;

$\frac{1}{b-a}\sum_{n=2}^{\infty}\frac{b^{n-1}-a^{n-1}}{z^n}$ for $|z|>|b|$; $\frac{1}{a-b}\left(\sum_{n=0}^{\infty}\frac{z^n}{b^{n+1}}+\sum_{n=1}^{\infty}\frac{a^{n-1}}{z^n}\right)$ for $|a|<|z|<|b|$.

577. $\frac{1}{z-2}+i\sum_{n=0}^{\infty}(-1)^n\frac{(2+i)^{n+1}-(2-i)^{n+1}}{5^{n+1}}(z-2)^n$ for $0<|z-2|<\sqrt{5}$;

$2\sum_{n=1}^{\infty}(-1)^n\frac{1}{z^{2n}}-\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}$ for $1<|z|<2$.

578. $-\frac{i}{4(z-i)}-\frac{1}{4(z-i)^2}+\sum_{n=0}^{\infty}\frac{(n+3)\,i^n(z-i)^n}{2^{n+4}}$ for $0<|z-i|<2$;

$\sum_{n=1}^{\infty}(-1)^n\frac{n}{z^{2n+2}}$ for $|z|>1$.

579. $\pm\left[z-\frac{1}{2}(a+b)+\sum_{n=1}^{\infty}\frac{c_{-n}}{z^n}\right]$ for $|z|>|b|$, where

$$c_{-(n-1)}=(-1)^n\left[\binom{\frac{1}{2}}{n}b^n+\binom{\frac{1}{2}}{n-1}\binom{\frac{1}{2}}{1}b^{n-1}a+\binom{\frac{1}{2}}{n-2}\binom{\frac{1}{2}}{2}b^{n-2}a^2+\ldots+\ldots\binom{\frac{1}{2}}{n}a^n\right].$$

580. $\sum_{n=1}^{\infty}\frac{c_{-n}}{z^n}+\sum_{n=0}^{\infty}c_nz^n$,

where $c_{-n}=\frac{(-1)^ni}{\sqrt{2}}\left[\binom{-\frac{1}{2}}{n}+\sum_{m=1}^{\infty}\binom{-\frac{1}{2}}{n+m}\binom{-\frac{1}{2}}{m}2^{-m}\right]$,

$c_n=2^{-n}c_{-n}$ $(n=1,2,\ldots)$, $c_0=\frac{i}{\sqrt{2}}\left[1+\sum_{m=1}^{\infty}\binom{-\frac{1}{2}}{m}2^{-m}\right]$.

581. $\frac{1}{2}+z+z^2+\sum_{n=1}^{\infty}\frac{1}{(n+2)!z^n}$ for $0<|z|<\infty$.

582. $\sum_{n=0}^{\infty}(-1)^n\frac{1}{n!(z-1)^n}$ for $0<|z-1|<\infty$;

$$1-\frac{1}{z}+\sum_{n=2}^{\infty}c_{-n}z^{-n} \text{ for } |z|>1,$$

where $c_{-n}=-1+\sum_{k=1}^{n-1}\frac{(-1)^{k+1}}{(k+1)!}\binom{n-1}{k}$ $(n=2,3,\ldots)$.

583. $\sum_{n=0}^{\infty}c_nz^n+\sum_{n=1}^{\infty}c_{-n}z^{-n}$, where $c_n=c_{-n}$

$$=\sum_{k=0}^{\infty}\frac{1}{k!(n+k)!}\ (n=0,1,2,\ldots).$$

584. $\sum_{n=0}^{\infty}c_{2n}z^{2n}+\sum_{n=0}^{\infty}c_{-2n}z^{-2n}$ where $c_{2n}=c_{-2n}$

$$=(-1)^n\sum_{k=0}^{\infty}\frac{1}{(2k+1)!\,(2n+2k+1)!}\ (n=0,1,2,\ldots).$$

585. $-\sum_{n=0}^{\infty} \frac{\sin\left(1+\frac{n\pi}{2}\right)}{n!\,(z-1)^n}$ for $0 < |z-1| < \infty$;

$-\sin 1 - \frac{\cos 1}{z} + \frac{\sin 1 - 2\cos 1}{2!\,z^2} + \frac{6\sin 1 - 5\cos 2}{3!\,z^3} + \ldots$ for $|z| > 1$.

586. $\frac{1}{z} + \sum_{n=1}^{\infty} (-1)^n \frac{2^{2n}B_{2n}}{(2n)!} z^{2n-1}$ where the B_{2n} are the Bernoulli numbers (see problem **515**), for $0 < |z| < 2\pi$; $\frac{3}{z} + 2\sum_{n=1}^{\infty} \frac{\pi^{2n}}{z^{2n+1}}$

$$+ \sum_{n=1}^{\infty} \left[(-1)^n \frac{2^{2n}B_{2n}}{(2n)!} + \frac{2}{\pi^{2n}}\right] z^{2n-1} \text{ for } \pi < |z| < 2\pi.$$

587. $\sum_{n=1}^{\infty} \frac{b^n - a^n}{nz^n}$ for $|z| > \max(|a|, |b|)$.

588. $\sum_{n=2}^{\infty} \frac{c_{-n}}{z^n}$ where $c_{-2k} = -i\left(2^{2k-1} - \frac{2^{2k-3}}{3} + \frac{2^{2k-5}}{5} - \ldots\right.$

$$\left. \ldots + (-1)^{k+1}\frac{2}{2k-1}\right),$$

$c_{-(2k+1)} = 2c_{-2k}$ $(k = 1, 2, \ldots)$, for $|z| > 2$; $i\tan^{-1}\frac{1}{2} \cdot \sum_{n=0}^{\infty} \frac{z^n}{2^n} + \sum_{n=1}^{\infty} \frac{c_{-n}}{z^n}$,

where $c_{-1} = 2i\tan^{-1} 1/2$, $c_{-2k} = 2^{2k} i\left(\tan^{-1} 1/2 - \sum_{m=0}^{k-1} \frac{(-1)^m}{(2m+1)\,2^{2m+1}}\right)$,

$c_{-(2k+1)} = 2c_{-2k}$ $(k = 1, 2, \ldots)$, for $1 < |z| < 2$.

589. (1) Yes; (2) Yes; (3) No (the point $z = 1$ is not an isolated singularity); (4) No; (5) No; (6) No; (7) No (in any ring round the point $z = 0$, the function is not eontinuous); (8) No; (9) Yes; (10) Yes, if α is an integer or zero; no in all the remaining eases.

590. (1) No; (2) Yes, both branches can be expanded; (3) No; (4) Yes, all three branches have expansions; (5) No; (6) Two branches of the four determined by the conditions $\sqrt{(1+\sqrt{1})} = \pm\sqrt{2}$ have expansions; (7) No; (8) No; (9) No; (10) Yes, all six branches have expansions; (11) No; (12), (13) and (14) Yes, all the branches have expansions; (15) No; (16) No; (17) All tne branehes have expansions except for two determined by the values $\sin^{-1}\sqrt{2}/2 = \pi/4$.

594. Onto the whole plane with a cut along the ray $e^{-i\alpha t}$, $\frac{1}{4} \leqslant t < \infty$.

595. Solution. If $f(z) = z + c_2 z^2 + \ldots$, then denoting by w_0 the value not assumed by the function $f(z)$ in the unit circle, we eonsider the function

$f(z)\Big/\left(1-\frac{f(z)}{w_0}\right)$. It is single valued in the circle $|z|<1$ (why?) and has the expansion $\frac{f(z)}{1-\frac{f(z)}{w_0}}=z+\left(c_2+\frac{1}{w_0}\right)z^2+\ldots$ By the theorem of problem 593: $|c_2|\leqslant 2$ and $\left|c_2+\frac{1}{w_0}\right|\leqslant 2$. Hence $|w_0|\geqslant\frac{1}{4}$.

597. $z=0$, $z=\pm 1$ are simple poles, $z=\infty$ is a regular point (a zero of the third order).

598. $z=(1\pm i)/\sqrt{2}$, $z=(-1\pm i)/\sqrt{2}$, are simple poles; $z=\infty$ is a regular point.

599. $z=1$ is a pole of the second order; $z=\infty$ is a pole of the third order.

600. $z=0$ is a pole of the first order; $z=\pm 2i$ are poles of the second order; $z=\infty$ is a regular point (a zero of the fifth order).

601. $z=\pm i$ are simple poles; $z=\infty$ is an essential singularity.

602. $z=\infty$ is an essential singularity.

603. $z=\infty$ is an essential singularity.

604. $z=2k\pi i$ ($k=\pm 1,\ \pm 2,\ \ldots$) are simple; $z=\infty$ is a limit point of poles.

605. $z=0$ is a pole of the second order; $z=2k\pi i$ ($k=\pm 1,\ \pm 2,\ \ldots$) are simple poles; $z=\infty$ is a limit point of poles.

606. $z=(2k+1)\pi i$ ($k=0,\pm 1,\pm 2,\ \ldots$) are simple poles; $z=\infty$ is a limit point of poles.

607. $z=k\pi i$ ($k=0,\pm 1,\pm 2,\ \ldots$) are simple poles; $z=\infty$ is a limit point of poles.

608. $z=0$ is an essential singularity; $z=\infty$ is a regular point.

609. $z=0$ is an essential singularity; $z=\infty$ is a simple pole.

610. $z=1$ is an essential singularity; $z=\infty$ is a regular point.

611. $z=0$ is an essential singularity; $z=\infty$ is an essential singularity.

612. $z=1$ is an essential singularity; $z=2k\pi i$ ($k=0,\pm 1,\pm 2,\ \ldots$) are poles of the first order; $z=\infty$ is a limit point of poles.

613. $z=k\pi$ ($k=0,\pm 1,\pm 2,\ \ldots$) are simple poles; $z=\infty$ is a limit point of poles.

614. $z=0$ is a pole of the second order; $z=\infty$ is an essential singularity.

615. $z=(2k+1)\frac{\pi}{2}$ ($k=0,\pm 1,\pm 2,\ \ldots$) are simple poles; $z=\infty$ is a limit point of poles.

616. $z=(2k+1)\frac{\pi}{2}$ ($k=0,\pm 1,\pm 2,\ldots$) are poles of the secoud order; $z=\infty$ is a limit point of poles.

617. $z=0$ is a pole of tho third order; $z=k\pi$ ($k=\pm 1,\ \pm 2,\ \ldots$) are simple poles; $z=\infty$ is a limit point of poles.

618. $z=k\pi$ ($k=\pm 1,\pm 2,\ \ldots$) are simple poles; $z=\infty$ is a limit point of poles.

619. $z=k\pi$ ($k=0,\pm 1,\pm 2,\ \ldots$) are simple poles; $z=\infty$ is a limit point of poles.

620. If $a\neq m\pi+\pi/2$ ($m=0,\pm 1,\pm 2,\ \ldots$), then $z=2k\pi+a$ and $z=(2k+1)\pi-a$ ($k=0,\pm 1,\pm 2,\ \ldots$) aro simple poles; if $a=m\pi+\pi/2$

then for m even, $z = 2k\pi + \frac{\pi}{2}$ and for m odd, $z = (2k+1)\pi + \frac{\pi}{2}$ are poles of the second order; $z = \infty$ is in each case the limit point of the poles.

621. If $a \neq m\pi$ $(m = 0, \pm 1, \pm 2, \ldots)$ then $z = (2k+1)\pi \pm a$ $(k = 0, \pm 1, \pm 2, \ldots)$ are simple poles; if $a = m\pi$ then for m odd $z = 2k\pi$, and for m even $z = (2k+1)\pi$ are poles of the second order, in each case $z = \infty$ is the limit point of the poles.

622. $z = 1$ is an essential singularity, $z = \infty$ is a regular point (a zero of the first order).

623 and **624.** $z = \frac{1}{k\pi}$ $(k = \pm 1, \pm 2, \ldots)$ are simple poles; $z = 0$ is the limit point of the poles; $z = \infty$ is a simple pole.

625. $z = 0$ is an essential singularity; $z = \infty$ is a regular point (a zero of the first order).

626. $z = 0$ is an essential singularity; $z = \infty$ is an essential singularity.

627. $z = \frac{1}{k\pi}$ $(k = \pm 1, \pm 2, \ldots)$ are essential singularities; $z = 0$ is the limit point of the essential singularities; $z = \infty$ is an essential singularity.

628. $z = 2/(2k+1)\pi$ $(k = 0, \pm 1, \pm 2, \ldots)$ are essential singularities; $z = 0$ is the limit point of the essential singularities; $z = \infty$ is a regular point.

629. $z = 1/k\pi$ $(k = \pm 1, \pm 2, \ldots)$ are essential singularities; $z = 0$ is the limit point of the essential singularities; $z = \infty$ is an essential singularity.

630. $z = 2/(2k+1)\pi$ $(k = 0, \pm 1, \pm 2, \ldots)$ are essential singularities; $z = 0$ is the limit point of the essential singularities; $z = \infty$ is a regular point.

631. (1) The point $z = \infty$ is a pole of order $k = \max(n, m)$, if $n \neq m$; if, however, $n = m$, then $z = \infty$ is either a pole of order $k \leqslant n$, or a regular point; (2) A pole of order $n-m$, if $n > m$ and a regular point if $n \leqslant m$; if $n < m$, then $z = \infty$ is a zero of order $m-n$; (3) A pole of order $n+m$.

633. Examples: (1) z^2; (2) $\frac{1}{z^2} + z$; (3) $1/(z^n - 1)$.

634. (1) $\frac{a}{z-\alpha}$ $(a \neq 0)$ or $az+b$ $(a \neq 0)$;

(2) $\frac{a}{(z-\alpha)^n}$ $(a \neq 0)$ or $a_0 + a_1 z + \ldots + a_n z^n$ $(a_n \neq 0)$;

(3) $\frac{1}{z^2} + C$;

(4) $\frac{a_0 + a_1 z + \ldots + a_{n+m} z^{n+m}}{z^n}$ $(a_0 \neq 0,\ a_{n+m} \neq 0)$;

(5) $\frac{a_0 + a_1 z + \ldots + a_n z^n}{(z-\alpha_1)(z-\alpha_2)\ldots(z-\alpha_n)}$ $(\alpha_k \neq \alpha_l$ for $k \neq l$ and at least one of the numbers a_m different from zero) or $\frac{a_0 + a_1 z + \ldots + a_n z^n}{(z-\alpha_1)(z-\alpha_2)\ldots(z-\alpha_{n-1})}$ $(a_n \neq 0,\ \alpha_k \neq \alpha_l$ for $k \neq l)$.

636. (1) z_0 is a removable singularity; (2) z_0 is a pole of order n, if $\phi(z)$ is single-valued in the neighbourhood of the point z_0, and a pole of order nm, if $\phi(z)$ is m-valued in the neighbourhood of this point; (3) z_0 is an essential singularity.

637. (1) The point z_0^* is a pole of order n if γ' is a rectilinear segment, and a regular point of multiplicity n, if γ' is a circular arc, that is, $f(z) - f(z_0^*)$

$= (z-z_0^*)^n \phi(z)$ where $\phi(z)$ is analytic in the neighbourhood of the point z_0^* and $\phi(z_0^*) \neq 0$. If $z_0^* = \infty$, then this condition is written in the form $f(z) - f(\infty) = z^{-n}\phi(z)$, where $\phi(z)$ is analytic at infinity and $\phi(\infty) \neq 0$;

(2) z_0^* is an essential singularity.

639. (1) -1; (2) 0; (3) 0; (4) 0.

640. (1) An essential singularity at $z = \infty$; the exceptional value is 0 (and ∞!); $e^z \to 0$ if $x \to -\infty$ ($e^z \to \infty$, if $x \to +\infty$);

(2) An essential singularity at $z = 0$; the exceptional value 0 (and ∞); $e^{\frac{1}{z}} \to 0$, if $z \to 0$, for example, along the path $y = 0$, $x < 0$ ($e^{\frac{1}{z}} \to \infty$ as $z \to 0$, along the path $y = 0$, $x > 0$);

(3) An essential singularity at $z = 0$; there are no exceptional values (not counting ∞); $\cos\frac{1}{z} \to \infty$ for $x = 0$, $y \to 0$;

(4) An essential singularity at $z = \infty$; the exceptional values are i and $-i$; $\tan z \to i$, if $y \to +\infty$, and $\tan z \to -i$, if $y \to -\infty$;

(5) An essential singularity at $z = \infty$; the exceptional value is -1; $\tan^2 z \to -1$, if $y \to \pm\infty$.

643. For example, $f(z) = \dfrac{1}{(1+z)^2} = 1-2z+3z^2-\ldots$; $s_{2n} = -n$, $s_{2n+1} = n+1$.

650. $\varrho = 0$.

651. $\varrho = n$, $\sigma = a$.

652. $\varrho = 1$, $\sigma = 3$.

653. $\varrho = 1$, $\sigma = 3$.

654. $\varrho = 3$, $\sigma = 2$.

655. $\varrho = 2$, $\sigma = 1/5$.

656. $\varrho = 1$, $\sigma = 1$.

657. $\varrho = 1$, $\sigma = 1$.

658. $\varrho = 1$, $\sigma = 1/2$.

659. $\varrho = 1/2$, $\sigma = 1$.

660. $\varrho = \dfrac{1}{2^m}$, $\sigma = 1$.

SOLUTION. $\displaystyle\sum_{n=0}^{\infty} \frac{z^{2n}}{(2^2.n)!} = \frac{1}{2}(\cos\sqrt{z} + \cos i\sqrt{z})$,

whence $\displaystyle\sum_{n=0}^{\infty} \frac{z^n}{(2^2.n)!} = \frac{1}{2}(\cos\sqrt[4]{z} + \cos i\sqrt[4]{z})$.

Similarly $\displaystyle\sum_{n=0}^{\infty} \frac{z^{2n}}{(2^3.n)!} = 1/4\ (\cos\alpha\sqrt[4]{z} + \cos\alpha^2\sqrt[4]{z} + \cos\alpha^3\sqrt[4]{z} + \cos\alpha^4\sqrt[4]{z})$,

where $\alpha = \sqrt[4]{-1}$, whence $\displaystyle\sum_{n=0}^{\infty} \frac{z^n}{(2^3.n)!} = \frac{1}{4}\sum_{k=1}^{4} \cos\alpha^k \sqrt[8]{z}$ and so on.

661. $\varrho = \infty$.

662. SOLUTION. It is easily seen that it is sufficient to consider values of $z > 0$.

Then $\dfrac{\int_0^1 e^{zt^2}dt}{e^z} < 1$; on the other hand, if $0 < \alpha < 1$, then $\dfrac{\int_0^1 e^{zt^2}dt}{e^{\alpha z}}$

$$= \int_0^1 e^{z(t^2-\alpha)}\,dt \geqslant \int_{\sqrt{\alpha}}^1 e^{z(t^2-\alpha)}\,dt \to \infty \text{ if } z \to \infty.$$

Consequently, $\varrho = 1$, $\sigma = 1$.

664. $\varrho^* = \max(\varrho_1, \varrho_2)$.

665. (1) $\varrho^* = \varrho$, $\sigma^* \leqslant \sigma_1 + \sigma_2$; (2) $\varrho^* = \varrho$, $\sigma^* = \max(\sigma_1, \sigma_2)$.

666. (1) $\varrho^* \leqslant \varrho$, $\sigma^* \leqslant 2\sigma$; (2) $\varrho^* \leqslant \varrho$, $\sigma^* \leqslant \sigma$.

667. SOLUTION. Let $M(r)$, ϱ, σ be the maximum of $|f(z)|$ on the circle $|z| = r$, the order and type of the function $f(z)$, and let $M_1(r)$, ϱ_1 and σ_1 be the corresponding characteristics for the function $f'(z)$. From the equality $f(z) = \int_0^z f'(t)\,dt + f(0)$ it follows that $M(r) \leqslant rM_1(r) + |f(0)|$ and, consequently, $\varrho \leqslant \varrho_1$. Since $f'(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)d\zeta}{(\zeta - z)^2}$ where as Γ can be taken the circle with centre at z and radius $r + \delta$ ($\delta > 0$ is arbitrary), it follows that $M_1(r) \leqslant \frac{M(2r+\delta)\,(r+\delta)}{\delta^2}$, that is $\varrho_1 \leqslant \varrho$, and hence, $\varrho_1 = \varrho$. Hence also from the inequalities given above we conclude that $\sigma_1 = \sigma$. Another possible method of proof is based on the theorem given on page 99.

669. $\varrho = 1$, $\sigma = 1/e$.

670. $\varrho = a$, $\sigma = \infty$.

671. $\varrho = a$, $\sigma = 0$.

672. $\varrho = 0$.

673. $\varrho = 0$.

674. $\varrho = 1$, $\sigma = 1$.

675. $\varrho = 1$, $\sigma = 2$.

676. $h(\phi) = \cos\phi$.

677. $h(\phi) = |\sin\phi|$.

678. $h(\phi) = |\sin\phi|$.

679. $h(\phi) = |\cos\phi|$.

680. $h(\phi) = \cos n\phi$.

681. $h(\phi) = \cos\phi$, if $\cos\phi \geqslant 0$; $h(\phi) = 0$, if $\cos\phi < 0$.

682. $h^*(\phi) = h(\phi)$, if $h(\phi) > 0$; $h^*(\phi) = 0$, if $h(\phi) < 0$; $h^*(\phi) \leqslant 0$, if $h(\phi) = 0$.

CHAPTER VI

683. The annulus $1/2 < |z| < 1$.
684. The exterior of the unit circle ($|z| > 1$).
685. $|z| < 1$.
686. The half-plane Re $z < -1$.
687. The real axis.
688. The whole plane except for the points $z = 0, \pm 1, \pm 2, \ldots$
689. $|z| > 1$.
690. $|z| < 1$.
691. The whole plane except for the unit circle ($|z| \neq 1$).
692. The whole plane except for the points $z = 4^{\frac{1}{n}} e^{\frac{(2k+1)\pi i}{n}}$ $(k, n = 1, 2, \ldots)$.

693. SOLUTION. If the series $\sum_{n=1}^{\infty} a_n$ converges, then for $|z| < 1$ the convergence of the series $\sum_{n=1}^{\infty} \frac{a_n z^n}{1-z^n}$ is obvious and from the identity $\sum_{n=1}^{\infty} \frac{a_n z^n}{1-z^n} = -\sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} \frac{a_n \left(\frac{1}{z}\right)^n}{1-\left(\frac{1}{z}\right)^n}$ it follows that the series also converges for $|z| > 1$.

If the series $\sum_{n=1}^{\infty} a_n$ diverges then the series $\sum_{n=1}^{\infty} a_n z^n$ has the radius of convergence $R \leqslant 1$. For $|z| > 1$ the divergence of the series $\sum_{n=1}^{\infty} \frac{a_n z^n}{1-z^n}$ follows from the fact that if this were not so the series $\sum_{n=1}^{\infty} \frac{a_n}{1-z^n}$ would converge, and consequently also the series $\sum_{n=1}^{\infty} \left(\frac{a_n}{1-z^n} - \frac{a_n z^n}{1-z^n} \right) = \sum_{n=1}^{\infty} a_n$. If, however $|z| < 1$, then the modulus q of the ratio of the general terms of the series $\sum_{n=1}^{\infty} a_n z^n$ and $\sum_{n=1}^{\infty} \frac{a_n z^n}{1-z^n}$ is contained within tho limits $1 - |z| \leqslant q \leqslant 2$ and consequently both series converge or diverge together.

694. (1) $\sum_{n=1}^{\infty} b_n z^n$, where $b_n = \sum a_p$, the summation extending to those indices p, which are divisors of the number n, including 1 and n. The radius of convergence $R = \min(r, 1)$, where r is the radius of convergence of the series $\sum_{n=1}^{\infty} a_n z^n$.

695. $\sum_{n=0}^{\infty} a_n (z-2)^n$, where $a_n = \frac{(-1)^n}{n!} \sum_{k=1}^{\infty} \frac{(\log k)^n}{k!}$ $\left(a_0 = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}\right)$; $R = 1$.

696. $1/2$, if $|z| < 1$, and $-1/2$, if $|z| > 1$.

697. $\frac{z}{(1-z)^2}$, if $|z| < 1$, and $\frac{1}{(1-z)^2}$, if $|z| > 1$.

698. z, if $|z| < 1$, and 1, if $|z| > 1$.

699. $z/(z-1)$, if $|z| < 1$, and $1/(z-1)$, if $|z| > 1$.

701. (1) and (2). Uniformly convergent in every circle $|z| \leqslant r < 1$ and in every domain $|z| \geqslant R > 1$;

(3) Converges uniformly on the whole real axis; diverges elsewhere.

703. Uniformly convergent on the circle $|z| = 1$; diverges everywhere else.

704. Uniformly convergent in any half-plane $\operatorname{Re} z \geqslant \delta$, where $\delta > 0$.

705. Uniformly convergent in any half-plane $\operatorname{Re} z \geqslant 1+\delta$, where $\delta > 0$.

706. Uniformly convergent on the real axis; diverges everywhere else.

707. Uniformly convergent on every segment $[2k\pi+\varepsilon,\ 2(k+1)\pi - \varepsilon]$ of the real axis ($k = 0, 1, 2, \ldots$).

708. No.

715. $x_c = x_a = -\infty$.

716. $x_c = -\infty$, $x_a = 1$.

717. $x_c = -\infty$, $x_a = +\infty$.

718. $x_c = 0$, $x_a = +\infty$.

719. $x_c = 0$, $x_a = 1$.

720. $x_c = x_a = -1$.

721. $x_c = x_a = +\infty$.

724. $x_c = 0$; it diverges at all the points of the boundary.

725. $x_c = x_a = -2$; it diverges at all the points of the boundary.

726. $x_c = x_a = 0$; it converges (non-absolutely) at the points $z = (2k+1)\pi i$ ($k = 0, \pm 1, \pm 2, \ldots$) and diverges at the remaining points of the boundary.

727. $x_c = x_a = 0$; absolutely convergent at all the points of the boundary.

728. $x_c = x_a = 0$; converges non-absolutely at all the points of the boundary.

736. The integral converges uniformly in every strip $0 < \alpha \leqslant \operatorname{Re} z \leqslant A < \infty$.

737. The integral converges uniformly in every half-plane $\operatorname{Re} z \geqslant \alpha > 0$.

738. The integral converges uniformly in every strip $\alpha \leqslant \operatorname{Re} z \leqslant 2 - \alpha$, where $\alpha > 0$.

739. The integral converges uniformly in every strip $\alpha \leqslant \operatorname{Re} z \leqslant 1 - \alpha$, where $\alpha > 0$.

740 and **741.** The integral converges uniformly in any closed interval of the real axis not containing the coordinate origin.

742. The integral converges uniformly in the half-plane $\operatorname{Im} z \geqslant 0$ with the semicircle $|z| < r$ deleted, where r is any arbitrarily small positive number.

743. The integral converges uniformly in the intervals $0 < \alpha \leqslant z \leqslant \beta < 1$ and $1 < \gamma \leqslant z < \infty$.

744. An example is: $f(t) = e^{t^2}$ for $n < t < n + e^{-n^3}$ $(n = 1, 2, \ldots)$ and $f(t) = 0$ for all other values of t.

745. $x_c = x_a = 0$.
746. $x_c = x_a = -\infty$.
747. $x_c = x_a = +\infty$.
748. $x_c = -\infty$; $x_a = 1$.
749. $x_c = -\infty$; $x_a = +\infty$.
750. $x_c = -1$; $x_a = +\infty$.
751. $x_c = 0$; $x_a = 1$.

752. Divergent at all the points of the boundary.

753. Absolutely convergent.

754. Convergent at the point $z = 0$, convergent (non-absolutely) at the remaining points of the boundary.

755. Non-absolutely convergent at all the points of the boundary.

765. SOLUTION. On evaluating the integrals present in the obvious inequalities

$$\int_0^{\frac{\pi}{2}} \sin^{2n+1} x\,dx < \int_0^{\frac{\pi}{2}} \sin^{2n} x\,dx < \int_0^{\frac{\pi}{2}} \sin^{2n-1} x\,dx,$$

we obtain $\left[\frac{(2n)!!}{(2n-1)!!}\right]^2 \frac{1}{2n+1} < \frac{\pi}{2} < \left[\frac{(2n)!!}{(2n-1)!!}\right]^2 \frac{1}{2n}$. In order to prove Wallis's formula it remains to establish that the difference between the extreme members of this set of inequalities tends to zero as $n \to \infty$.

767. (1) No; (2) Yes.

769. (1) Divergent; (2) Diverges (to zero); (3) Convergent; (4) Convergent.

770. Converges non-absolutely.

771. Divergent.

772. Convergent if $p > \frac{1}{2}$, the convergence being absolute if $p > 1$; divergent if $p \leqslant 1/2$.

773. Absolutely convergent if $p > 1$; divergent if $p \leqslant 1$.
774. Absolutely convergent.
776. $|z| < 1$.
777. $|z| < 2$.
778. $|z| < \infty$.
779. $|z| > 1$.
780. $|z| < 1/e$.
781. $|z| < \infty$.
782. $|z| < \infty$.
783. $|z| < \infty$.
784. $|z| < \infty$.

791. SOLUTION. (1) From the series $\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \ldots$ by subtracting the series for $2^{-s}\zeta(s)$, we obtain $(1-2^{-s})\,\zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \ldots$; the terms $\frac{1}{n^s}$, for which n is divisible by 2 are absent from the right-hand side of this equation. Similarly $(1-2^{-s}) \times (1-3^{-s})\,\zeta(s) = 1 + \frac{1}{5^s} + \frac{1}{7^s} + \ldots$, the terms $\frac{1}{n^s}$, for which n is divisible by 2 or by 3 being absent from the right-hand side. Generally

$$(1-p_1^{-s})\,(1-p_2^{-s}) \ldots (1-p_m^{-s})\,\zeta(s) = 1 + \sum \frac{1}{n^s}, \qquad (1)$$

the summation on the right of (1) extending to those indices n (greater than unity), which are not divisible by any of the numbers $p_1, p_2, \ldots, p_m$. It is easy to prove that for $\operatorname{Re} s \geqslant 1+\delta$ $(\delta > 0)$ the sum of the series on the right hand side of (1) tends to zero as $m \to \infty$, and, consequently, $\zeta(s) \prod_{m=1}^{\infty} (1-p_m^{-s}) = 1$.

(2) Since it follows from the test of absolute convergence (see problem **768**) that the product $\prod_{m=1}^{\infty} (1-p_m^{-s})$ converges for $\operatorname{Re} s \geqslant 1+\delta$, the function $\zeta(s)$ has no zeros for $\operatorname{Re} s > 1$.

792. SOLUTION. It follows from the proof of the preceding problem that for any $\delta > 0$ we have $\prod_{n=1}^{\infty} (1-p_n^{-(1+\delta)}) = \dfrac{1}{\zeta(1+\delta)}$. From this it is easily concluded that $\lim_{\delta \to 0} \prod_{n=1}^{\infty} (1-p_n^{-(1+\delta)}) = 0$. Since $(1-p_n^{-1}) < (1-p_n^{-(1+\delta)})$, it is obvious that the product $\prod_{n=1}^{\infty} (1-p_n^{-1})$ diverges (to zero), and consequently, the series $\sum_{n=1}^{\infty} p_n^{-1}$ also diverges.

CHAPTER VII

793. $\operatorname{res}[f(z)]_{z=\pm 1} = -1/2$; $\operatorname{res}[f(z)]_{z=0} = 1$; $\operatorname{res}[f(z)]_{z=\infty} = 0$.

794. $\operatorname{res}[f(z)]_{z=i} = -1/4$; $\operatorname{res}[f(z)]_{z=-i} = \dfrac{i}{4}$; $\operatorname{res}[f(z)]_{z=\infty} = 0$.

795. $\operatorname{res}[f(z)]_{z=-1} = (-1)^{n+1}\dfrac{(2n)!}{(n-1)!\,(n+1)!}$; $\operatorname{res}[f(z)]_{z=\infty}$

$$= (-1)^n \frac{(2n)!}{(n-1)!\,(n+1)!}.$$

796. $\operatorname{res}[f(z)]_{z=0} = 1$; $\operatorname{res}[f(z)]_{z=\pm 1} = -1/2$; $\operatorname{res}[f(z)]_{z=\infty} = 0$.

797. $\operatorname{res}[f(z)]_{z=0} = 0$; $\operatorname{res}[f(z)]_{z=1} = 1$; $\operatorname{res}[f(z)]_{z=\infty} = -1$.

798. $\operatorname{res}[f(z)]_{z=-1} = 2\sin 2$; $\operatorname{res}[f(z)]_{z=\infty} = -2\sin 2$.

799. $\operatorname{res}[f(z)]_{z=0} = 1/9$; $\operatorname{res}[f(z)]_{z=3i} = -\dfrac{1}{54}(\sin 3 - i\cos 3)$;

$\operatorname{res}[f(z)]_{z=-3i} = -\dfrac{1}{54}(\sin 3 + i\cos 3)$; $\operatorname{res}[f(z]_{z=\infty} = \dfrac{1}{27}(\sin 3 - 3)$.

800. $\operatorname{res}[f(z)]_{z=\frac{2k+1}{2}\pi} = -1$ $(k = 0, \pm 1, \pm 2, \ldots)$.

801. $\operatorname{res}[f(z)]_{z=k\pi} = (-1)^k$ $(k = 0, \pm 1, \pm 2, \ldots)$.

802. $\operatorname{res}[f(z)]_{z=k\pi} = 0$ $(k = 0, \pm 1, \pm 2, \ldots)$.

803. $\operatorname{res}[f(z)]_{z=k\pi} = -1$ $(k = 0, \pm 1, \pm 2, \ldots)$.

804. (1) $\operatorname{res}[f(z)]_{z=2} = \operatorname{res}[f(z)]_{z=\infty} = 0$;

(2) $\operatorname{res}[f(z)]_{z=2} = -\operatorname{res}[f(z)]_{z=\infty} = -143/24$.

805. $\operatorname{res}[f(z)]_{z=0} = -\operatorname{res}[f(z)]_{z=\infty} = \displaystyle\sum_{n=0}^{\infty} \frac{1}{n!\,(n+1)!}$.

806. $\operatorname{res}[f(z)]_{z=0} = \operatorname{res}[f(z)]_{z=\infty} = 0$.

807. $\operatorname{res}[f(z)]_{z=-1} = -\operatorname{res}[f(z)]_{z=\infty} = -\cos 1$

808. $\operatorname{res}[f(z)]_{z=-3} = -\operatorname{res}[f(z)]_{z=\infty}$

$$= -\sin 2\left[\sum_{n=1}^{\infty} \frac{4^{2n}}{(2n-1)!\,(2n)!} + \sum_{n=0}^{\infty} \frac{4^{2n+1}}{(2n)!\,(2n+1)!}\right].$$

809. $\operatorname{res}[f(z)]_{z=0} = 1/2$; $\operatorname{res}[f(z)]_{z=\frac{2k\pi i}{n}} = \dfrac{1}{2k\pi i}$ $(k = \pm 1, \pm 2, \ldots)$.

810. $\operatorname{res}[f(z)]_{z=0} = 0$, if $n < 0$, and also if $n > 0$ is odd; $\operatorname{res}[f(z)]_{z=0}$

$= \dfrac{(-1)^{\frac{n}{2}}}{(n+1)!}$, if $n = 0$ or $n > 0$ is even; $\operatorname{res}[f(z)]_{z=\infty} = -\operatorname{res}[f(z)]_{z=0}$.

811. $\operatorname{res}[f(z)]_{z=\frac{1}{k\pi}} = (-1)^{k+1}\dfrac{1}{k^2\pi^2}$ $(k = \pm 1, \pm 2, \ldots)$;

$$\operatorname{res}[f(z)]_{z=\infty} = \frac{2}{\pi^2}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2} = -\frac{1}{6}.$$

812. $\operatorname{res}[f(z)]_{z=k^2\pi^2} = (-1)^k 2k^2\pi^2$ $(k = 1, 2, \ldots)$.

813. $\operatorname{res}[f(z)]_{z=0} = 0$, if n is odd, $\operatorname{res}[f(z)]_{z=0} = (-1)^{k+1}\dfrac{2^{2k}(2^{2k}-1)}{(2k)!}B_{2k}$, if $n = 2k$ $(k = 0, \pm 1, \pm 2, \ldots)$, where B_{2k} are the Bernoulli numbers (see problem **515**); $\operatorname{res}[f(z)]_{z=\left(k+\frac{1}{2}\right)\pi} = -\dfrac{1}{\left(k+\dfrac{1}{2}\right)^n\pi^n}$ $(k = 0, \pm 1, \pm 2 \ldots)$.

814. 1; -1.

815. 0; 2.

816. $-2e^{2k\pi ai}$, if $\sqrt{1} = 1$ and $\operatorname{Log} 1 = 2k\pi i$; 0 for the branch defined by the value $\sqrt{1} = -1$.

817. $\pm\dfrac{(a-b)^2}{8}$.

818. (1) $\alpha-\beta$ (for all the branches); (2) $e^{\alpha}-e^{\beta}$ (for all the branches).

819. (1) $2^{k\pi i} + \dfrac{1}{2.3!} - \dfrac{1}{4.5!} + \ldots$, if $\operatorname{Log} 1 = 2k\pi i$;

(2) $-\dfrac{1}{2!} + \dfrac{1}{3.4!} - \dfrac{1}{5.6!} + \ldots$ (for all the branches).

820. $\operatorname{res}[f(z)]_{z=0} = k\pi$, if $\operatorname{Tan}^{-1} 0 = k\pi$; $\operatorname{res}[f(z)]_{z=\infty} = -\dfrac{(2k+1)\pi}{2}$, if $\operatorname{Tan}^{-1}\infty = \dfrac{(2k+1)\pi}{2}$.

821. $\operatorname{res}[f(z)]_{z=0} = 0$, if $n \geqslant 0$; $\operatorname{res}[f(z)]_{z=0} = \operatorname{Log}\dfrac{\alpha}{\beta}$, if $n = -1$, and $\operatorname{res}[f(z)]_{z=0} = \dfrac{1}{n+1}(\alpha^{n+1}-\beta^{n+1})$, if $n \leqslant -2$; $\operatorname{res}[f(z)]_{z=\infty} = \dfrac{1}{n+1}(\alpha^{n+1} - \beta^{n+1})$, if $n \geqslant 0$; $\operatorname{res}[f(z)]_{z=\infty} = -2k\pi i$ if $n = -1$ and $\operatorname{Log} 1 = 2k\pi i$, and $\operatorname{res}[f(z)]_{z=\infty} = 0$, if $n \leqslant -2$.

822. $-2c_0c_1$.

823. (1) $A\phi(a)$; (2) $c_{-1}\phi(a) + \dfrac{c_{-2}\phi'(a)}{1!} + \ldots + \dfrac{c_{-k}\phi^{(k-1)}(a)}{(k-1)!}$.

824. (1) n; (2) $-n$.

825. (1) $n\phi(a)$; $-n\phi(a)$.

826. $\dfrac{A}{\phi'(a)}$.

827. AB.

828. $\displaystyle\sum_{n=1}^{k}(-1)^n\frac{n\bar{c}_{-n}(\beta^*-a)^{n+1}}{R^{2n}}$.

829. $-\dfrac{\pi i}{\sqrt{2}}$.

830. $-2\pi i$.

831. $-\pi i/121$.

832. πi.

833. $-2\pi i/9$.

834. 1.

835. 0.

836. $\frac{2^{n+1}}{(n+1)!}$, if $n \geqslant -1$, and 0 if $n < -1$.

837. $\frac{f(0)}{g(0)} + \sum_{k=1}^{n} \frac{f(a_k)}{a_k g'(a_k)}$.

838. 0, if $r < 1$; ± 1 if $r > 1$ (the sign depends on the choice of the branch of the integrand).

839. $\frac{1}{4}\sqrt{(1+\sqrt{2})}$.

840. $\pi/(1+a)$.

841. $e^{\pi/2}/(1+e^{\pi})$.

842. $2\pi/\sqrt{(a^2-1)}$.

843. $2\pi a/(a^2-b^2)^{3/2}$.

844. $(2a+b)\pi/[a(a+b)]^{3/2}$.

845. $2\pi/(1-a^2)$ if $|a| < 1$; $2\pi/(a^2-1)$ if $|a| > 1$; 0 (the principal value) if $|a| = 1$, $a \neq \pm 1$ (for $a = \pm 1$ the principal value does not exist).

846. $\pi(a^6+1)/(1-a^2)$, if $|a| < 1$; $\frac{\pi(a^6+1)}{a^6(a^2-1)}$, if $|a| > 1$; $\frac{\pi}{2}\frac{1-a^{12}}{a^6(a^2-1)}$ (the principal value), if $|a| = 1$, $a \neq \pm 1$ (for $a = \pm 1$ the principal value does not exist).

847. $2\pi/n!$, if $n \geqslant 0$; 0, if $n < 0$.

848. $\pi i \operatorname{sign} a$ (for $a = 0$ the principal value of the integral equals 0).

849. $-2\pi i \operatorname{sign} \operatorname{Im} a$.

851. $-\pi/27$.

852. $\pi/4a$.

853. $\frac{1.3.5 \ldots (2n-3)}{2.4.6 \ldots (2n-2)}\frac{\pi}{2}$, if $n > 1$; $\pi/2$, if $n = 1$.

854. $\frac{\pi}{ab(a+b)}$.

855. $\pi/\sqrt{2}$.

857. $(-1)^{k-1}/(2ih-z)^k$.

858. (1) $\frac{\pi}{3e^3}(\cos 1 - 3\sin 1)$; (2) $\frac{\pi}{3e^3}(3\cos 1 + \sin 1)$.

860. $\frac{\pi e^{-|ab|}}{2|b|}$.

861. $\frac{\pi}{2} e^{-|ab|} \operatorname{sign} a$.

859. $\frac{\pi}{2e^4}(2\cos 2 + \sin 2)$.

863. πi, if $t > 0$; 0, if $t = 0$; $-\pi i$, if $t < 0$.

864. $\pi(2\sin 2 - 3\sin 3)$.

865. $\frac{\pi}{5}(\cos 1 - 1/e^2)$.

866. $\frac{\pi}{3}[\sin a + e^{-(a\sqrt{3})/2}(\sin(a/2) + \sqrt{3}\cos(a/2))]$.

867. $\pi(e^{-|ab|} - 1/2) \operatorname{sign} a$.

868. $\frac{\pi}{2b^2}(1 - e^{-|ab|}) \operatorname{sign} a$.

869. $\frac{\pi}{4b^4}[2 - (2+|ab|)e^{-|ab|}] \operatorname{sign} a$.

870. $\pi(|b| - |a|)$.

871. $\pi/2$.

872. $3\pi/8$.

873. (1) $\frac{\Gamma(p)\cos\frac{\pi p}{2}}{a^p}$; (2) $\frac{\Gamma(p)\sin\frac{\pi p}{2}}{a^p}$; in order to verify the correctness

the answer for $-1 < p < 0$, it is sufficient to note that for these values of p the integral converges and the function in the answer is analytic.

874. $\frac{1}{p}\Gamma\left(\frac{1}{p}\right)\cos\frac{\pi}{2p}$.

875. $\frac{1}{p}\Gamma\left(\frac{1}{p}\right)\sin\frac{\pi}{2p}$.

876. $\frac{1}{p-1}\Gamma\left(\frac{1}{p}\right)\cos\pi/2p$ (for $p = 1$ the integral is equal to $\pi/2$).

878. $\pi/\sin p\pi$.

880. $\pi/2\cos\frac{\pi p}{2}$.

881. $\frac{\pi(1-p)}{4\cos\frac{\pi p}{2}}$ (for $p = 1$ the integral equals $1/2$).

882. $\frac{\pi}{\sin p\pi}\frac{\sin p\lambda}{\sin\lambda}$, if $\lambda \neq 0$, and $\frac{p\pi}{\sin p\pi}$ if $\lambda = 0$.

884. $\pi\cot p\pi$.

885. $\pi\cot p\pi$.

886. $\frac{\pi p(1-p)}{2^{3-p}\sin p\pi}$.

887. $\frac{\pi}{\sin p\pi}\left(2^{\frac{p}{2}}\cos\frac{p\pi}{4}-1\right)$.

888. $\frac{\pi}{\sin p\pi}\left[2^p\left(1-\frac{p}{2}\right)-1\right]$.

889. $\frac{\pi}{\sin p\pi}\left(\sin\frac{p\pi}{2}+\cos\frac{p\pi}{2}-1\right)$.

890. $\pi\sqrt[3]{4}/\sqrt{3}$.

891. If a does not belong to the interval $(-1,1)$, then $I = -\frac{\pi}{\sqrt{(a^2-1)}}$, where $\sqrt{(a^2-1)} > 0$ for $a > 1$ (in the plane with a cut along the segment $[-1, 1]$ the quantity $\sqrt{(a^2-1)}$ is single valued) for $a = \pm e^{i\alpha}$, $I = \pm\frac{\pi}{\sqrt{(2\sin\alpha)}}e^{i\left(\frac{3\pi}{4}-\frac{\alpha}{2}\right)}$; for $a = iy$, $I = \frac{\pi i}{\sqrt{(1+y^2)}}\operatorname{sign} y$; for $a < -1$, $I = \frac{\pi}{\sqrt{(a^2-1)}}$; for $-1 < a < 1$, $I = 0$ (the principal value).

892. If b does not belong to the interval $(0,1)$, then $I = \frac{\pi}{\sin p\pi}b^{p-1}(b-1)^{-p}$, where $(b-1)^{-p} > 0$, $b^{p-1} > 0$ for $b > 1$; if $0 < b < 1$, then $I = -\pi b^{p-1}\times(1-b)^{-\alpha}\cot p\pi$ (the principal value).

893. $I_0 = \frac{\pi}{2\sqrt{2}}$, $I_{2k} = \frac{\pi}{2}\left(\frac{1}{\sqrt{2}}-S_{2k}\right)$, $I_{2k+1} = -\frac{\pi}{2}\left(\frac{1}{\sqrt{2}}-S_{2k+1}\right)$ where S_ν is the sum of ν terms of the series $1-\frac{1}{2}+\frac{1.3}{2.4}-\ldots+(-1)^\nu\frac{1.3\ldots(2\nu-1)}{2.4\ldots 2\nu}$.

894. $I_0 = \frac{\pi}{\sqrt{3}}$, $I_n = \frac{\pi}{\sqrt{3}}\frac{1.4\ldots(3n-2)}{3.6\ldots 3n}$ $(n = 1,2,\ldots)$.

895. $\pi/\left(n\sin\frac{\pi}{n}\right)$.

896. $\frac{\pi}{2|a|}\log|a|$.

897. $\frac{\pi}{8|a|}(\pi^2+4\log^2|a|)$.

898. $\frac{\pi}{2|a|^3\sqrt{(2|a|)}}\left(\frac{3}{2}\log|a|-1-\frac{3\pi}{4}\right)$.

899. $-\pi$.

900. $\frac{\pi}{2}\log 2$.

901. $\pi^2/4$.

903. (1) $\dfrac{1}{1-a}+\dfrac{1}{\log a}$ (for $a=1$, $I=1/2$); (2) $\dfrac{\pi}{2a\left(\log^2 a+\dfrac{\pi^2}{4}\right)}-\dfrac{1}{1+a^2}$.

904. $\dfrac{1}{2n+1}\left\{\dfrac{\pi}{2a}\sum_{k=0}^{2n}(-1)^k\dfrac{2k+1}{\log^2 a+\left(k+\dfrac{1}{2}\right)^2\pi^2}-\dfrac{1}{1+a^2}\right\}$.

905. $\dfrac{1}{2n}\left\{\dfrac{1}{1+a^2}+\dfrac{\pi}{2a}\sum_{k=0}^{2n-1}(-1)^{k+1}\dfrac{2k+1}{\log^2 a+\left(k+\dfrac{1}{2}\right)^2\pi^2}\right\}$.

907. $\dfrac{\pi}{2}\tanh\dfrac{\pi a}{2}$. **909.** $1/\left(2\cos\dfrac{a}{2}\right)$.

908. $\dfrac{\pi^2 e^{\pi a}}{(e^{\pi a}+1)^2}$.

910. $\dfrac{\pi}{a}\log(1+a)$, if $0<a<1$; $\dfrac{\pi}{a}\log\dfrac{1+a}{a}$ if $a>1$.

913. (1) $t^n/n!$; (2) $\dfrac{\log^n t}{n!}$, if $t>1$; 0, if $t<1$; if $t=1$, then $I=0$ for $n>1$ and $I=1/2$ (the principal value) for $n=1$.

914. $e^{at}t^n/n!$. **916.** $\cos t$.

915. $\sin t$. **917.** $t-\sin t$.

918. $\dfrac{e^{at}}{(b-a)(c-a)}+\dfrac{e^{bt}}{(a-b)(c-b)}+\dfrac{e^{ct}}{(a-c)(b-c)}$.

919. $\dfrac{1}{n!}\left(1-\dfrac{1}{t}\right)^n$, if $t\geqslant 1$; 0, if $t\leqslant 1$.

922. (1) $\dfrac{e^{-t}}{\sqrt{(\pi t)}}$; (2) $\dfrac{e^{-it}}{\sqrt{(\pi t)}}$. **924.** $e^{-t}\operatorname{erf}\sqrt{t}$.

925. $\sin t$, if $t<\pi$; 0, if $t\geqslant\pi$.

923. $\operatorname{erf}\sqrt{t}$, where $\operatorname{erf} u=\dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^u e^{-x^2}dx$.

926. 1, if $0<t<a$; 0, if $t=a$; -1, if $a<t<2a$; $-\dfrac{1}{2}$, if $t=2a$; 0, if $t>2a$.

927. $n+1$, if $na<t<(n+1)a$; $n+\dfrac{1}{2}$, if $t=na$; $(n=0, 1, 2, \ldots)$.

928. $1-\operatorname{erf}\dfrac{x}{2\sqrt{t}}$ (see the answer to problem **923**).

929. $\frac{1}{a}+2\sum_{n=1}^{\infty}(-1)^n e^{-\frac{n^2\pi^2 t}{a^2}}\frac{\sin\frac{n\pi r}{a}}{n\pi r}$. **931.** $\frac{2\sinh t}{t}$.

930. $-Ei(-t)$, where $Ei(t)=\int_{-\infty}^{t}\frac{e^u}{u}du$. **932.** $\frac{2\pi i}{a}$.

933. 0, if $b<0$; $\cosh^{-1}\frac{b}{a}$, if $b>0$.

936. $\frac{1}{\sqrt{(z^2+1)}\,[z+\sqrt{(z^2+1)}]^n}$ ($\sqrt{(z^2+1)}>0$ for $z>0$).

937. (1) $\frac{1}{\sqrt{(a^2-b^2)}}$ if $a>b$; 0, if $a<b$; (2) 0, if $a>b$; $\frac{1}{\sqrt{(b^2-a^2)}}$, if $a<b$.

938. $\frac{\pi}{2}J_0(i)|a|\sqrt{(t^2-b^2)}$.

939. If the real part of even one pole is positive, then $\lim_{t\to\infty} f(t)=\infty$; if the real parts of all the poles are negative, then $\lim_{t\to\infty} f(t)=0$; if some poles are situated on the imaginary axis and all the remainder have negative real parts, then $f(t)$ oscillates as $t\to\infty$, the amplitude of the oscillation increasing without limit if there is even one pole on the imaginary axis of higher order than the first, and remains bounded if all the poles on the imaginary axis are simple. If there is only one pole on the imaginary axis at the coordinate origin, then $f(t)\to\infty$, if the pole is multiple, and $f(t)\to \operatorname{res}[e^{zt}\phi(z)]_{z=0}$, if the pole is simple.

941. $f(t)\sim(at-3)/a^4$.

942. $f(t)\sim\frac{i\omega}{\sqrt{(2ai\omega-\omega^2)}}e^{t\omega t-\sqrt{(2ai\omega-\omega^2)}}$.

945. Solution. Let us represent the integral in the form $\int_{-x}^{x}\frac{e^{-x-t}}{t}dt+\int_{x}^{\infty}\frac{e^{-x-t}}{t}dt$. Since on asymptotic expansion in negative powers of x, $e^{-x}\sim 0$, then by the solution of problem **944** it follows that the second integral is asymptotically equal to zero. By the definition of the principal value of an integral,

$$\int_{-x}^{x}\frac{e^{-x-t}}{t}dt=e^{-x}\lim_{\varepsilon\to 0}\left[\int_{-x}^{-\varepsilon}\frac{e^{-t}}{t}dt+\int_{\varepsilon}^{x}\frac{e^{-t}}{t}dt\right]$$

$$=e^{-x}\lim_{\varepsilon\to 0}\int_{\varepsilon}^{x}\frac{e^{-t}-e^{t}}{t}dt=e^{-x}\int_{0}^{x}\frac{e^{-t}-e^{t}}{t}dt$$

(in the last integral the integrand is continuous). Moreover,

$$\int_0^x \frac{e^{-t}-e^t}{t}\,dt = \int_0^1 \frac{e^{-t}-e^t}{t}\,dt + \int_1^x \frac{e^{-t}}{t}\,dt + \int_1^x \frac{e^t}{t}\,dt \sim O(1) + \int_1^x \frac{e^t}{t}\,dt.$$

Integrating by parts, we obtain

$$\int_1^x \frac{e^t}{t}\,dt = C + e^x\left(\frac{1}{x} + \frac{1}{x^2} + \ldots + \frac{(n-1)!}{x^n}\right) + n!\int_1^x \frac{e^t}{t^{n+1}}\,dt,$$

where $C = C(n)$ is a constant. It remains to establish, that

$$\lim_{x\to\infty} x^n e^{-x}\int_1^x \frac{e^t}{t^{n+1}}\,dt = 0,$$

and this is easily proved by l'Hôpital's rule.

948. SOLUTION. Let us consider the integral $\int_C e^{z^2-t^2}\,dt$, where the contour C is represented in Fig. 69 (for $\operatorname{Re} z > 0$). This integral is

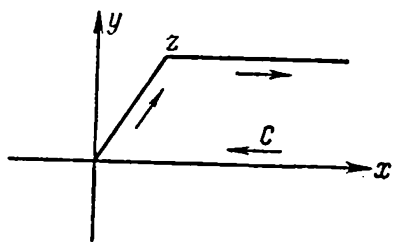

FIG. 69

equal to zero and hence ($z = x+iy$)

$$\int_0^z e^{z^2-t^2}dt = \int_0^\infty e^{z^2-t^2}dt - \int_x^\infty e^{z^2-(t+iy)^2}dt.$$

The first integral on the right-hand side equals $\frac{\sqrt{\pi}}{2}e^{z^2}$; let us write the second in the form

$$-\frac{1}{2}\int_x^\infty \frac{1}{t+iy}\,de^{z^2-(t+iy)^2} = \frac{1}{2z} - \frac{1}{2}\int_x^\infty \frac{e^{z^2-(t+iy)^2}}{(t+iy)^2}\,dt.$$

Repeating the integration by parts we obtain the required expansion. For the remainder we have the estimate

$$\left|\frac{1.3\ldots(2n-1)}{2^n}\int_x^\infty \frac{e^{z^2-(t+iy)^2}}{(t+iy)^{2n}}\,dt\right| < \frac{1.3\ldots(2n-1)}{2^n}\int_x^\infty \frac{e^{x^2-t^2}}{|t+iy|^{2n}}\,dt.$$

Again integrating by parts, we obtain $\int_x^\infty \frac{e^{x^2-t^2}}{|(t+iy|^{2n}}\,dt < \frac{1}{2|z|^{2n}x}$, whence it follows that the expansion is asymptotic. The case Re $z < 0$ is treated similarly. If however, Re $z = 0$, then

$$\left|\int_0^\infty \frac{e^{z^2-(t+iy)^2}}{(t+iy)^{2n}}\,dt\right| < \int_0^\infty \frac{e^{-t^2}}{y^{2n}}\,dt = \frac{\sqrt{(\pi)}}{2y^{2n}}.$$

949. $f(t) \sim \frac{1}{\omega}\sin\left(\omega t+\frac{\pi}{4}\right) - \frac{1}{\omega^2\pi} \times$

$$\times\left[\frac{\Gamma\left(\frac{3}{2}\right)}{t^{3/2}} - \frac{\Gamma\left(\frac{7}{2}\right)}{\omega^2 t^{7/2}} + \ldots + (-1)^n \frac{\Gamma\left(2n+\frac{3}{2}\right)}{\omega^{2n}t^{2n+3/2}} + \ldots\right];$$

for small t, $f(t) = 2\sqrt{(t/\pi)}\left[1-\frac{(2\omega t)^2}{1.3.5} + \frac{(2\omega t)^4}{1.3.5.7.9} - \ldots\right] \approx 2\sqrt{(t/\pi)}$.

951. $f(t) \sim 1+\frac{1}{\pi}\sum_0^\infty (-1)^n \frac{\Gamma\left(3n+\frac{3}{2}\right)}{t^{3n+\frac{3}{2}}}$;

for small t, $f(t) = \frac{t^{\frac{3}{2}}}{\Gamma\left(\frac{5}{2}\right)} - \frac{t^3}{\Gamma(4)} + \frac{t^{\frac{9}{2}}}{\Gamma\left(\frac{11}{2}\right)} - \ldots \approx \frac{4t^{\frac{3}{2}}}{3\sqrt{\pi}}$.

952. 1.
953. 0.
954. 4.
956. 1; 3.
957. 0.
958. 1.
959. n.

961. Solution. Since the sequence of functions $f_n(z)$ converges to the function $e^{1/z}$ everywhere except at the point $z = 0$, then for any small circle K with centre at a point $z \neq 0$ and not containing the coordinate origin either within it or on its boundary, for a given $\epsilon > 0$ it is possible to find an N such that the inequality $|f_n(z)-e^{1/z}| < \epsilon$ is satisfied for all the points of the small circle K for $n > N$. It remains to choose $\varepsilon < \min_{z\in C}|e^{1/z}|$, where C is the circumference of the circle K, and apply Rouché's theorem.

Remark. The assertion of the problem follows directly from Hurwitz's theorem (see, for example, [2, Chapter VIII, § 2].

965. 0.
966. 2; 1.

967. One root in each quadrant.

968. Two roots in each of the second and third quadrants.

970. In the domain $\beta > 0$, $\alpha > +\sqrt{\beta}$ (the domain I in Fig. 70) $m = 0$; in the domain $\beta > 0$, $\alpha < +\sqrt{\beta}$ (the domain II) $m = 2$; in the domain $\beta < 0$ (the domain III) $m = 1$.

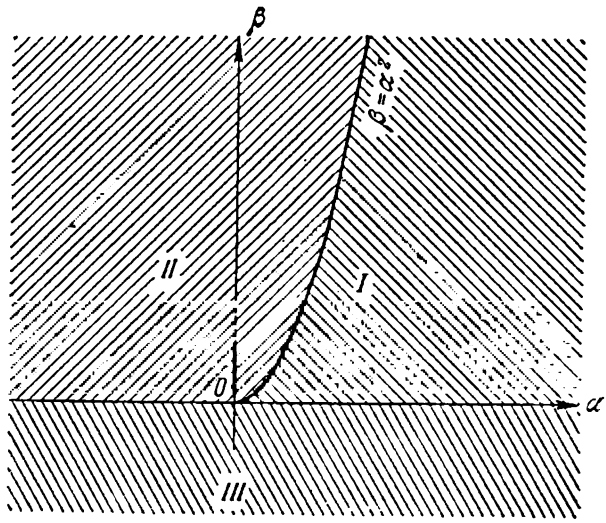

FIG. 70

971. In the domain $\alpha > 0$, $\beta > \frac{1}{\alpha}$ (the domain I in Fig. 71) $m = 0$; in the domain where either $\alpha \leqslant 0$, or $\alpha > 0$, $\beta < \frac{1}{\alpha}$ (the domain II), $m = 2$.

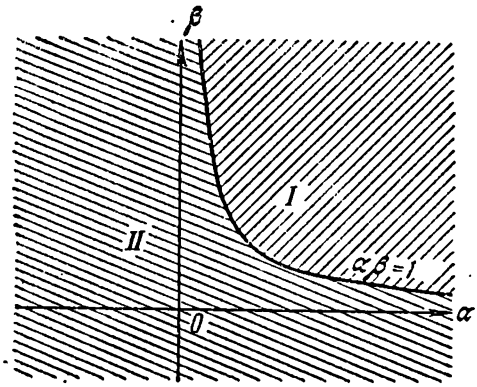

FIG. 71

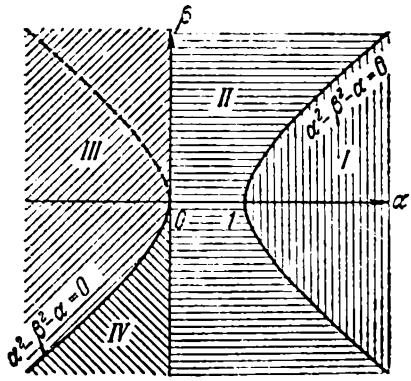

FIG. 72.

972. In the domain $\alpha > \frac{1}{2} + \sqrt{\left(\beta^2 + \frac{1}{4}\right)}$ (the domain I in Fig. 72) $m = 0$; in the domain $0 < \alpha < \frac{1}{2} + \sqrt{\left(\beta^2 + \frac{1}{4}\right)}$ (the domain II) $m = 2$; in the domain where either $\alpha < \frac{1}{2} - \sqrt{\left(\beta^2 + \frac{1}{4}\right)}$, or $\frac{1}{2} - \sqrt{(\beta^2+4)} \leqslant \alpha < 0$,

$\beta > 0$ (the domain III), $m = 1$; in the domain $\frac{1}{2} - \sqrt{\left(\beta^2 + \frac{1}{4}\right)} < \alpha < 0$, $\beta < 0$ (the domain IV) $m = 3$.

974. The domain containing the positive semi-axis a and bounded by the lines $a+b=0$ and $\begin{cases} a = -\dfrac{t\cos\tau t}{\sin\tau t}, \\ b = \dfrac{t}{\sin\tau t}, \end{cases} \quad 0 \leqslant t \leqslant \dfrac{\pi}{2}$.

975. The domain situated in the first quadrant and bounded by the lines $b = 0$ and $\begin{cases} a = \tan\tau t, \\ b = \dfrac{t^2}{\cos\tau t}, \end{cases} \quad 0 \leqslant t \leqslant \dfrac{\pi}{2\tau}$.

976. The finite domain bounded by the segment $b = 0$, $0 \leqslant a \leqslant \dfrac{\pi}{2\tau}$ and the arc $\begin{cases} a = t\sin\tau t, \\ b = t^2\cos\tau t, \end{cases} \quad 0 \leqslant t \leqslant \dfrac{\pi}{2\tau}$.

986. $z = a + \dfrac{w}{1!}f(a) + \dfrac{w^2}{2!}\dfrac{\mathrm{d}}{\mathrm{d}a}\{[f(a)]^2\} + \ldots + \dfrac{w^n}{n!}\dfrac{\mathrm{d}^{n-1}}{\mathrm{d}a^{n-1}}\{[f(a)]^n\} + \ldots$

987. $z = \dfrac{1}{2}w + \sum_{n=2}^{\infty}(-1)^{n-1}\dfrac{n(n+1)\ldots(2n-2)}{2^{2n-1}.n!}w^n$;

$z = -2 - \dfrac{1}{2}w + \sum_{n=2}^{\infty}(-1)^n\dfrac{n(n+1)\ldots(2n-2)}{2^{2n-1}.n!}w^n$.

988. $z = a + \dfrac{w}{1!}\dfrac{1}{2}(a^2-1) + \dfrac{w^2}{2!}\dfrac{1}{2^2}\dfrac{\mathrm{d}}{\mathrm{d}a}[(a^2-1)^2] + \ldots$

$\ldots + \dfrac{w^n}{n!}\dfrac{1}{2^n}\dfrac{\mathrm{d}^{n-1}}{\mathrm{d}a^{n-1}}[(a^2-1)^n] + \ldots$

990. (1) $z = \sum_{n=1}^{\infty}\dfrac{n^{n-1}a^{n-1}}{n!}w^n$; (2) $\mathrm{e}^{bz} = b\sum_{n=0}^{\infty}\dfrac{(an+b)^{n-1}}{n!}w^n$.

991. $z = a + \sum_{n=1}^{\infty}\dfrac{w^n}{n!}\dfrac{\mathrm{d}^{n-1}}{\mathrm{d}a^{n-1}}(\sin^n a)$.

992. SOLUTION. The function $w = (z - \pi/2)/(\sin z)$ is analytic in tho disk $\left|z - \dfrac{\pi}{2}\right| < r$, if $r < \pi/2$ and in this disk there are no zeros other than $z = \pi/2$. On the circumference of this circle $|w| \geqslant 2r/(\mathrm{e}^r + \mathrm{e}^{-r})$, hence the disk $|w| < \varrho$ of radius $\varrho < 2r/(\mathrm{e}^r + \mathrm{e}^{-r})$ is mapped one-one onto a corresponding neighbourhood of the point $z = \pi/2$ and the expansion of $z(w)$ converges in this circle. The function $2r/(\mathrm{e}^r + \mathrm{e}^{-r})$ has a maximum at $r = r^*$, where e^{2r^*}

$= \dfrac{r^*+1}{r^*-1}$ ($r^* = 1{\cdot}19 \ldots < \pi/2$), and this maximum is equal to $\dfrac{2r^*}{e^{r^*}+e^{-r^*}}$ $= \sqrt{(r^{*2}-1)} = 0{\cdot}6627 \ldots$. Thus, the required radius of convergence is not less than 0·6627 ... At the same time the point w, at which $dw/dz = 0$, that is, $\tan z = z - \pi/2$, cannot be inside the circle of convergence of the expansion of $z(w)$. Putting $z - \dfrac{\pi}{2} = t$ we transform the equation $dw/dz = 0$ into the form $\cot t = -t$ or $e^{2it} = (it+1)/(it-1)$. Consequently, $it = r^*$ is a root of the equation $dw/dz = 0$. The corresponding value is given by $w = t/\cos t$ $= -2ir^*/(e^{r^*}+e^{-r^*})$, whence $|w| = \dfrac{2r^*}{e^{r^*}+e^{-r^*}} = \sqrt{(r^{*2}-1)} = 0{\cdot}6627 \ldots$ Thus, this point is located on the circumference of the circle of convergence and the radius of convergence is consequently equal to $\sqrt{(r^{*2}-1)} = 0{\cdot}6627\ldots$

1013. $\pi^2/(\sin^2 \pi a)$.

1014. $\dfrac{\pi^2 \cot \pi a}{\sin \pi a}$.

1015. $\pi^2/8$.

1016. $\dfrac{1}{2a^2}(1+\pi a \coth \pi a)$.

1017. $\dfrac{1}{2a^2}\left(1+\dfrac{\pi a}{\sinh \pi a}\right)$.

1018. $\pi^3/32$.

1019. $\dfrac{\pi}{2}\dfrac{\sin ab}{\sin \pi a}$.

CHAPTER VIII

1024. (1) $F^+(z) = (z-a)^n$, $F^-(z) \equiv 0$; (2) $F^+(z) \equiv 0$, $F^-(z) = -1/(z-a)^n$; (3) $F^+(z) = 1/(z-a)^n$, $F^-(z) \equiv 0$. In all three cases an integral of Cauchy type is transformed into a Cauchy integral.

1025. (1) $F^+(z) = \phi(z) - \sum g_k\left(\frac{1}{z-a_k}\right)$, $F^-(z) = -\sum g_k\left(\frac{1}{z-a_k}\right)$;

(2) $F^+(z) = \sum g_k\left(\frac{1}{z-a_k}\right) + g(z)$,

$F^-(z) = -\phi(z) + \sum g_k\left(\frac{1}{z-a_k}\right) + g(z)$. $g_k\left(\frac{1}{z-a_k}\right)$ is the principal part of the expansion of the function $\phi(z)$ in the neighbourhood of the pole a_k; $g(z)$ is the principal part of the expansion of $\phi(z)$ in the neighbourhood of the point at infinity, in which the constant term is included.

1026. $F^+(z) = \dfrac{z+\log\dfrac{z-2}{z-3}}{z^2-4}$, $F^-(z) = -\dfrac{i}{z^2+1}$.

1027. $F^+(z) = \operatorname{eot} z - \dfrac{1}{z} - \dfrac{2z}{z^2-\pi^2}$, in particular, $F^+(0) = 0$, $F^+(\pi) = -\dfrac{3}{2\pi}$, $F^+(-\pi) = \dfrac{3}{2\pi}$; $F^-(z) = -\dfrac{1}{z} - \dfrac{2z}{z^2-\pi^2}$.

1028. $F^+(z) = \dfrac{z}{2(z+i)}$, $F^-(z) = -\dfrac{z}{2(z-i)}$.

1029. $F^+(z) = \dfrac{1}{2}\sum_{n=0}^{\infty}\dfrac{(a_n-ib_n)\,z^n}{R^n}$, $F^-(z) = -\dfrac{1}{2}\sum_{n=1}^{\infty}\dfrac{R^n(a_n+ib_n)}{z^n}$;

the limiting values are: $F^+(Re^{i\theta}) = \dfrac{1}{2}\sum_{n=0}^{\infty}(a_n-ib_n)\,e^{in\theta}$

$$= \frac{a_0}{4} + \frac{1}{2}\phi(Re^{i\theta}) + \frac{i}{2}\sum_{n=1}^{\infty}(-b_n\cos n\theta + a_n\sin n\theta);$$

$F^-(Re^{i\theta}) = -\dfrac{1}{2}\sum_{n=1}^{\infty}(a_n+ib_n)\,e^{in\theta}$

$$= \frac{a_0}{4} - \frac{1}{2}\phi(Re^{i\theta}) + \frac{i}{2}\sum_{n=1}^{\infty}(-b_n\cos n\theta + a_n\sin n\theta).$$

1030. (1) If the point z lies in the circle Q_k: $|z-k\pi| < \dfrac{\pi}{2}$, then $I_1(z)$

$= f(z-k\pi)$, $I_2(z) = (-1)^k f(z-k\pi)$. In particular, if $|z| < \pi/2$, then $I_1(z) = I_2(z) = f(z)$. If the point z lies outside all the closed $\bar{Q}_k$, then $I_1(z) = I_2(z) = 0$.

(2) Let Q_k be the disk $|z-k\pi| < \pi$. Then $I_1(z) = f(z-k\pi) + f[z-(k-1)\pi]$,
$I_2(z) = (-1)^k [f(z-k\pi) - f(z-(k-1)\pi)]$, if $z \in Q_{k-1}Q_k$;
$I_1(z) = f(z-k\pi) + f[z-(k+1)\pi]$,
$I_2(z) = (-1)^k [f(z-k\pi) - f(z-(k+1)\pi)]$, if $z \in Q_kQ_{k+1}$;
$I_1(z) = f(z-k\pi)$, $I_2(z) = (-1)^k f(z-k\pi)$, if z lies within the domain $Q_k - Q_{k-1}Q_k - Q_kQ_{k+1}$; $I_1(z) = 0$, $I_2(z) = 0$, if z lies within the complement to all the Q_k.

1031. $F(z) = \dfrac{1}{2\pi i} \log \dfrac{z-1}{z+1}$ † ; $F^{\pm}(\zeta) = \dfrac{1}{2\pi i} \log \dfrac{1-\zeta}{1+\zeta} \pm \dfrac{1}{2}$,

$F(\zeta) = \dfrac{1}{2\pi i} \log \dfrac{1-\zeta}{1+\zeta}$ $(-1 < \zeta < 1)$; $F(\pm i) = \pm \dfrac{1}{4}$, $F^{\pm}(0) = \pm \dfrac{1}{2}$,
$F(0) = 0$.

1032. $F(z) = \dfrac{1}{2\pi i} \operatorname{Log} \dfrac{z+R}{z-R}$ — a single-valued branch in the z-plane with a cut along C is defined by the value $\operatorname{Log} 1 = 0$.

$\operatorname{Log} \dfrac{z+R}{z-R} = \log \left|\dfrac{z+R}{z-R}\right| + i\Delta_C \arg(\zeta - z)$, where $\Delta_C \arg(\zeta - z)$ is the increment of $\arg(\zeta - z)$ along C.

$$F^+(\zeta) = \frac{1}{2\pi i} \log \left|\frac{\zeta+R}{\zeta-R}\right| + \frac{3}{4}, \quad F^-(\zeta) = \frac{1}{2\pi i} \log \left|\frac{\zeta+R}{\zeta-R}\right| - \frac{1}{4},$$

$$F(\zeta) = \frac{1}{2\pi i} \log \left|\frac{\zeta+R}{\zeta-R}\right| + \frac{1}{4}; \quad F(0) = \frac{1}{2},$$

$F^+(iR) = 3/4$, $F^-(iR) = -1/4$, $F(iR) = 1/4$; $F'(0) = -i/\pi R$.

1033. $F(z) = \dfrac{1}{2\pi i} \operatorname{Log} \dfrac{z+R}{z-R}$ (a single-valued branch in the z-plane with a cut along C is defined by the value $\operatorname{Log} 1 = 0$; for $|z| > R$ it is identical with the similar branch of problem **1032**);

$$F^+(\zeta) = \frac{1}{2\pi i} \log \left|\frac{\zeta+R}{\zeta-R}\right| + 1/4, \quad F^-(\zeta) = \frac{1}{2\pi i} \log \left|\frac{\zeta+R}{\zeta-R}\right| - 3/4;$$

$$F(0) = -\frac{1}{2}, \quad F'(0) = -\frac{i}{\pi R}.$$

---

† At first glance the solution is obviously:

$$\frac{1}{2\pi i} \int_{-1}^{1} \frac{d\zeta}{\zeta - z} = \frac{1}{2\pi i} [\log(1-z) - \log(-1-z)] = \frac{1}{2\pi i} \log \frac{z-1}{z+1}.$$

However, it is necessary to verify that the final transformation really leads to the given branch of the logarithm, since the equation $\log z_2 - \log z_1 = \log z_2/z_1$, is not generally true. This remark must be borne in mind in what follows also.

1034. (1) 0, if $|z| < r$ or $|z| > R$, and $1/z^n$, if $r < |z| < R$; (2) $1/z^n$, if $\operatorname{Im} z > \pi$, and 0, if $\operatorname{Im} z < \pi$; (3) 0, if $|\operatorname{Im} z| < \pi$, and $-1/z^n$, if $|\operatorname{Im} z| > \pi$; (4) $\dfrac{1}{2\pi i z^n}\left[\operatorname{Log}\dfrac{R+z}{R-z} - 2\sum\limits_{k=1}^{m}\dfrac{z^{2k-1}}{(2k-1)R^{2k-1}}\right]$, if $m = \left[\dfrac{n}{2}\right]$, $\operatorname{Log}\dfrac{R+z}{R-z} = \log\left|\dfrac{R+z}{R-z}\right| + i\Delta_C\{\arg(\zeta-z) - \arg\zeta\}$ is the single-valued branch in the z-plane with a cut along C, defined by the value $\operatorname{Log} 1 = 0$. The limiting value of this branch on C has imaginary part $\pi/2$ on the left and $\left(-\dfrac{3\pi}{2}\right)$ on the right. This determines the limiting values $F^{\pm}(\zeta)$ on C; $F(0) = \dfrac{1}{2\pi i}\dfrac{1+(-1)^{n-1}}{nR^n}$; (5) the function $F(z)$ is the same as in the preceding section, only $\operatorname{Log}\dfrac{R+z}{R-z}$ is the single-valued branch in the z-plane with a cut along the semicircle C, with the same value $\operatorname{Log} 1 = 0$.

The limiting value of this branch on C has imaginary part $3\pi/2$ on the left and $\left(-\dfrac{\pi}{2}\right)$ on the right. This defines the limiting values of $F^{\pm}(\zeta)$ on C; $F(0) = \dfrac{1}{2\pi i}\dfrac{1+(-1)^{n-1}}{nR^n}$.

REMARK. In parts (4) and (5) the branches of $\operatorname{Log}\dfrac{R+z}{R-z}$ are identical in the disk $|z| < R$, generally they are different; thus, for example, at ∞ in the first case the branch has the value $-\pi i$ and in the second $+\pi i$. However, $F(\infty) = 0$.

1035. $\operatorname{Log}\dfrac{b-z}{a-z} = \log\left|\dfrac{b-z}{a-z}\right| + i\Delta_C \arg(\zeta - z)$ is the single-valued branch in the z-plane with a cut along C, defined by the value $\operatorname{Log} 1 = 0$ (the branches are defined in the same way in problems **1036–1040**).

1036. $b-a+z\operatorname{Log}\dfrac{b-z}{a-z}$.

1037. $\sum\limits_{k=1}^{n} A_{nk}z^{k-1} + z^n \operatorname{Log}\dfrac{b-z}{a-z}$, $A_{nk} = \dfrac{b^{n-k+1} - a^{n-k+1}}{n-k+1}$.

1038. $\sum\limits_{n=1}^{\infty}\sum\limits_{k=1}^{n} c_n A_{nk} z^{k-1} + \phi(z) \operatorname{Log}\dfrac{b-z}{a-z}$.

1039. $\dfrac{1}{z-z_0}\left(\operatorname{Log}\dfrac{b-z}{a-z} - \operatorname{Log}\dfrac{b-z_0}{a-z_0}\right)$.

1040. $\dfrac{1}{(z-z_0)^n}\left[\operatorname{Log}\dfrac{b-z}{a-z} - \operatorname{Log}\dfrac{b-z_0}{a-z_0} + \sum\limits_{k=2}^{n} A_k (z-z_0)^{k-1}\right]$,

$A_k = \dfrac{1}{k-1}\left[\dfrac{1}{(b-z_0)^{k-1}} - \dfrac{1}{(a-z_0)^{k-1}}\right]$, $F(z_0) = -A_{n+1}$.

1041. (1) $F^+(z) = \log(z+R)$, $F^-(z) = \log\left(1+\dfrac{R}{z}\right)$;

$F^+(\zeta) = \log 2R\cos\dfrac{\phi}{2} + i\dfrac{\phi}{2}$, $F^-(\zeta) = \log 2\cos\dfrac{\phi}{2} - i\dfrac{\phi}{2}$;

(2) $F^+(z) = \log(R-z)+\pi i$†, $F^-(z) = \log\left(1-\dfrac{R}{z}\right)$;

$F^+(\zeta) = \log 2R\left|\sin\dfrac{\phi}{2}\right| + i\dfrac{\pi+\phi}{2}$, $F^-(\zeta) = \log 2\left|\sin\dfrac{\phi}{2}\right| + i\dfrac{\pi-\phi}{2}$ (the sense of the angle $\phi = \arg\zeta$ corresponds to the conditions of the problem).

1042. (1) $F^+(z) = 0$, $F^-(z) = \log\left(1-\dfrac{1}{z}\right)$; $F^+(\zeta) = 0$,

$F^-(\zeta) = \log\left(1-\dfrac{1}{\zeta}\right)$; (2) $F^+(z) = \log\dfrac{z}{z-1}$, $F^-(z) = 0$; $F^+(\zeta) = \log\dfrac{\zeta}{\zeta-1}$, $F^-(\zeta) = 0$.

1043. $F(z) = \dfrac{1}{2\pi i}\sum\limits_{n=1}^{\infty}\dfrac{1}{nz^n}\left[\operatorname{Log}\dfrac{R+z}{R-z} - 2\sum\limits_{k=1}^{m}\dfrac{z^{2k-1}}{(2k-1)\,R^{2k-1}}\right]$ where $m = \left[\dfrac{n}{2}\right]$, and $\operatorname{Log}[(R+z)/(R-z)]$ has the same value as in problem **1034**, (4). This determines the limiting values $F^{\pm}(\zeta)$ on C. If $|z| < R$, then $F(z) = \sum\limits_{n=0}^{\infty} c_n z^n$, where $c_n = \dfrac{1}{2\pi i}\int\limits_C \log\dfrac{\zeta}{\zeta-1}\dfrac{d\zeta}{\zeta^{n+1}}$.

1044 $F^+(z) = -\dfrac{1}{\pi i}\left[2+\sqrt{z}\log\dfrac{1-\sqrt{z}}{1+\sqrt{z}}\right]$,

$F^-(z) = -\dfrac{1}{\pi i}\left[2+\sqrt{z}\log\dfrac{1-\sqrt{z}}{1+\sqrt{z}}\right] - \sqrt{z}$.

1045. $F^+(z) \equiv 0$, $F^-(z) = \operatorname{Log}\dfrac{z-b}{z-a}$ (the branch of the logarithm is determined by the conditions of the problem).

1046. $F^+(z) \equiv 1$, $F^-(z) = 1 - \sqrt{[(z-a)/(z-b)]}$.

1047. $F^+(z) = z-\lambda a-(1-\lambda)b$, $F^-(z) = z-\lambda a-(1-\lambda)b-(z-a)^{\lambda}(z-b)^{1-\lambda}$.

1048. $F^+(z) = \operatorname{Log}\dfrac{z-\zeta_0}{z-b}$, $F^-(z) = \operatorname{Log}\dfrac{z-\zeta_0}{z-a}$.

1050. $-x^2+x-3\cdot 5$.

1051. (1) $\dfrac{1}{2}\left(\log\dfrac{z}{z-1}\right)^2$; (2) $\dfrac{1}{2}\left(\log\dfrac{\tau}{1-\tau}\right)^2 - \dfrac{\pi^2}{2}$.

† If $\operatorname{Log} z$ and $\log z$ are the branches indicated in the conditions of the problem, then $\operatorname{Log} z = \log(-z)+\pi i$.

1053. If $|z|>1$ and $z\in C$, then $F(z)=\frac{1}{2\pi i}\log\frac{z}{z-1}\operatorname{Log}\frac{R+z}{R-z}+F_1(z)$, where $F_1(z)$ is an analytic function for $|z|>1$, and the branch of $\operatorname{Log}\frac{R+z}{R-z}$ is chosen just as in problem **1034**, (4). From this the behaviour of $F(z)$ at the points $\pm R$ at the ends of the arc C is obvious.

1059. $F(z)=-2z\log\log 1/r+2z\log\log 1/R$, $\frac{\partial F}{\partial x}=-2\log\log 1/r-2\frac{e^{i\phi}\cos\phi}{\log r}+2\log\log 1/R$, $\frac{\partial F}{\partial y}=-2i\log\log 1/r-2\frac{e^{i\phi}\sin\phi}{\log r}+2i\log\log 1/R$, $\frac{\partial F}{\partial\bar z}=\phi(z)=\frac{e^{2i\phi}}{\log 1/r}$, $\frac{\partial F}{\partial z}=-2\log\log\frac{1}{r}-\log\frac{1}{r}+2\log\log\frac{1}{R}$.

1063. $D(f)$ is the area of the domain G', onto which $f(z)$ maps the domain G.

1069. (1) $\log\left|\frac{R^2-\bar\zeta z}{R(z-\zeta)}\right|$; (2) $\log\left|\frac{z-\bar\zeta}{z-\zeta}\right|$; (3) $\log\left|\frac{e^{\pi z}-e^{\pi\bar\zeta}}{e^{\pi z}-e^{\pi\zeta}}\right|$.

1076. $u(z)=\frac{1}{2\pi}\int_0^{2\pi}u(\zeta)\frac{r^2-R^2}{R^2-2Rr\cos(\theta-\phi)+r^2}\,d\theta$,

$u(\infty)=\frac{1}{2\pi}\int_0^{2\pi}u(\zeta)\,d\theta$.

1078. (1) $f(z)=\phi(z)+\overline{\psi\left(\frac{R^2}{\bar z}\right)}$, $f_1(z)=-\overline{\phi\left(\frac{R^2}{\bar z}\right)}-\psi(z)$, where

$v(0)=\operatorname{Im}f(0)=\operatorname{Im}[\phi(0)+\overline{\psi(\infty)}]$;

(2) $f(z)=-i\phi(z)+\overline{i\psi\left(\frac{R^2}{\bar z}\right)}$, $f_1(z)=\overline{i\phi\left(\frac{R^2}{\bar z}\right)}+i\psi(z)$,

$v(0)=\operatorname{Im}f(0)=\operatorname{Im}[-i\phi(0)+\overline{i\psi(\infty)}]$.

1079. (1) $f(z)=z^n$, $f_1(z)=-R^{2n}/z^n$ (here and in the answers to problems **1080–1083** the value of $v(0)$ is taken as equal to zero).

1080. $f(z)=\frac{z^n}{R^{2n}}$, $f_1(z)=-1/z^n$.

1081. $f(z)=-\log\left(1-\frac{z}{R^2}\right)$ $(\log 1=0)$, $f_1(z)=\log(1-1/z)$.

1082. $f(z)=\frac{R}{\sqrt{(R^2-z)}}\left(f(0)=\frac{1}{R}\right)$, $f_1(z)=-\sqrt{[z/(z-1)]}$.

1083. $f(z)\equiv\log R$, $f_1(z)\equiv-\log R$.

1085. $u(z)=\frac{1}{\pi}\int_{-\infty}^{\infty}u(t)\frac{y\,dt}{(t-x)^2+y^2}$, $f(z)=\frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{u(t)dt}{t-z}+iC$

(for the existence of the first integral the piecewise continuity and boundedness of $u(t)$ in the whole interval $(-\infty,\infty)$ is sufficient, for the second, in

addition, for example, the function $u(t)$ must be of order $\frac{1}{|t|^\alpha}$ $(\alpha > 0)$ at infinity).

1086. $f(z) = u(z)+iv(z)$

$$= \frac{1}{2i}\int_{-\infty}^{\infty} u(t)\coth\frac{\pi(t-z)}{2}\,dt - \frac{1}{2i}\int_{-\infty}^{\infty} u_1(t)\tanh\frac{\pi(t-z)}{2}\,dt,$$

where $u_1(t) = u(t+i)$ (for the existence of the integrals it is sufficient, for example that $u(t)$ and $u_1(t)$ should decrease at infinity like $\frac{1}{|t|^{1+\alpha}}$, $\alpha > 0$).

1088. Circles inside the disk $|z| < 1$, tangential to the circle $|z| = 1$ at the point $e^{i\theta}$.

1090. Arcs of circles, connecting the points $e^{i\theta_1}$, $e^{i\theta_2}$ inside the circle $|z| < 1$.

1091. $\omega(z; a, b) = \frac{1}{\pi}\arg\frac{z-b}{z-a}$, $\omega(z; -\infty, b) = \frac{1}{\pi}\arg(z-b)$; $\omega(z;\ a, \infty)$ $= 1-\frac{1}{\pi}\arg(z-a)$. The geometrical meaning of these harmonic measures is the angle divided by π subtended by the segment or ray at the point z.

1092. $1-\frac{1}{\gamma}\arg z$ for the ray $\arg z = 0$ and $\frac{1}{\gamma}\arg z$ for the ray $\arg z = \gamma$.

1093. $\omega(z, \Delta) = \frac{2}{\pi}\arg\frac{z-R}{z+R} - 1 = 1-\frac{2}{\pi}\tan^{-1}\frac{2Ry}{R^2-|z|^2}$, $\omega(z, \Gamma)$ $= 1-\omega(z, \Delta)$. The level lines are circular arcs connecting the points $\pm R$, from the points of which the diameter Δ subtends an angle $\frac{\pi}{2}(1+\omega)$ in the case of $\omega(z, \Delta)$ and $\pi\left(1-\frac{\omega}{2}\right)$ in the case of $\omega(z, \Gamma)$.

1094. $\omega(z, \Gamma) = \frac{2}{\pi}\arg\frac{z-R}{z+R}$.

1095. $\omega(z, \Gamma) = \frac{2}{\pi}\arg\frac{\sqrt{z}-\sqrt{R}}{\sqrt{z}+\sqrt{R}}$.

1096. $\frac{\log|z|-\log r}{\log R-\log r}$ for $|z| = R$ and $\frac{\log|z|-\log R}{\log r-\log R}$ for $|z| = r$.

1099. $u(z) = \sum_{\nu=1}^{n} c_\nu\omega_\nu(z)$.

1108. $\sin n\phi$.

1109. $-\cos n\phi$.

1110. $\frac{a\sin\phi}{1-2a\cos\phi+a^2}$.

1111. $\arg(1-ae^{i\phi}) = -\sin^{-1}\frac{a\sin\phi}{\sqrt{(1-2a\cos\phi+a^2)}}$.

1112. $\frac{1}{2^{2m-1}}\sum_{k=0}^{m-1} C_{2m}^k\sin 2(m-k)\phi$.

1113. $\frac{\sin\frac{n\phi}{2}\sin\frac{(n+1)\phi}{2}}{\sin\phi/2}$.

1114. $-\frac{4}{\pi}\left(\sin\phi+\frac{\sin 3\phi}{3^2}+\frac{\sin 5\phi}{5^2}+\dots\right)$.

1115. $-2\log 2\cos\frac{\phi}{2}$.

1116. $-2\log 2\sin\frac{\phi}{2}$.

1122. $\tilde{h}(\phi)=\sin 2\phi$.

k	$\tilde{h}_k$ Approx.
0	0·00000
1	0·58779
2	0·95106
3	0·95106
4	0·58779

1123. $\tilde{h}(\phi)=\sin 3\phi$.

k	$\tilde{h}_k$ Approx.
0	0·00000
1	0·80903
2	0·95107
3	0·30902
4	−0·58779
5	−1·00001

1124. $\tilde{h}(\phi)=\sin 4\phi$.

k	$\tilde{h}_k$ Approx.
0	0·00000
1	0·95104
2	0·58777
3	−0·58778
4	−0·95104
5	0·00000

1125. $\tilde{h}(\phi)=-\cos\phi$.

k	$\tilde{h}_k$ Approx.
0	−1·00000
1	−0·95107
2	−0·80902
3	−0·58779
4	−0·30902
5	0·00000

1126. $\tilde{h}(\phi)=-\cos 3\phi$.

k	$\tilde{h}_k$ Approx.
0	−1·00002
1	−0·58780
2	0·30902
3	0·95108
4	0·80903
5	0·00000

1127. $\tilde{h}(\phi) = -\frac{8}{\pi^2}\left(\cos\phi + \frac{\cos 3\phi}{3^3} + \frac{\cos 5\phi}{5^3} + \ldots\right)$.

k	$\tilde{h}_k$ Approx.	$\tilde{h}_k$ Actual
0	−0·84925	−0·85247
1	−0·78768	−0·78556
2	−0·64004	−0·64039
3	−0·44958	−0·44940
4	−0·23107	−0·23120
5	−0·00000	−0·00000

1128. $\tilde{h}(\phi) = -\frac{4}{\pi}\left(\cos\phi - \frac{\cos 3\phi}{3^2} + \frac{\cos 5\phi}{5^2} - \ldots\right)$.

k	$\tilde{h}_k$ Approx.	$\tilde{h}_k$ Actual
0	−1·1868	−1·1662
1	−1·1240	−1·1345
2	−1·0611	−1·0815
3	−0·8430	−0·8585
4	−0·6248	−0·5694
5	0·0000	0·0000

1131. (2) $T[\phi(x)] = -i[\phi(x) - 2\phi_a(x) - \phi(\infty)]$,
$T[\psi(x)] = i[\psi(x) - 2\psi_b(x) - \psi(\infty)]$,
where $\phi_a(z)$ and $\psi_b(z)$ are the principal parts of the expansions of $\phi(z)$ and $\psi(z)$ in the neighbourhoods of the points a and b.

No. of problem	$\tilde{h}(x)$	$f(z)$	$f_1(z)$
1132	$\sin\lambda x$	$e^{i\lambda z}$	$-e^{-i\lambda z}$
1133	$-\cos\lambda x$	$-ie^{i\lambda z}$	$-ie^{-i\lambda z}$
1134	$-\frac{b}{x^2+b^2}$	$\frac{1}{z+bi}$	$-\frac{1}{z-bi}$
1135	$\frac{x(1+x^2)}{(\sqrt{2})(1+x^4)}$	$\frac{\sqrt{2}-iz}{(\sqrt{2})(1-z^2-i(\sqrt{2})z)}$	$-\frac{\sqrt{2}+iz}{(\sqrt{2})(1-z^2+i(\sqrt{2})z)}$
1136	$\mathrm{Im}\,\frac{1}{(x-a+ib)^n}$	$\frac{1}{(z-a+ib)^n}$	$-\frac{1}{(z-a-ib)^n}$
1137	$\mathrm{Im}\,\frac{1}{(x-a+ib)^n}$	$\frac{1}{(z-a+ib)^n}$	$-\frac{1}{(z-a-ib)^n}$

1141. If $h(x) \equiv 1$ in the interval $(-1, 1)$, then $\tilde{h}(x) = -\frac{1}{\pi}\log\frac{x-1}{x+1}$ for $|x| > 1$; if $h(x) = 1-|x|$, then $\tilde{h}(x) = \frac{1}{\pi}\left[x\log\frac{x^2-1}{x^2} - \log\frac{x-1}{x+1}\right]$; if $h(x) = \operatorname{sign} x$, then $\tilde{h}(x) = \frac{1}{\pi}\log\frac{x^2}{x^2-1}$.

1142. By the rectangle formula: $\tilde{h}(1) = 0{\cdot}60716$; $\tilde{h}(2) = 0{\cdot}34003$; $\tilde{h}(3) = 0{\cdot}20116$; $\tilde{h}(4) = 0{\cdot}14595$; $\tilde{h}(5) = 0{\cdot}11524$. By the asymptotic expansion: $\tilde{h}(2) = 0{\cdot}34$; $\tilde{h}(3) = 0{\cdot}2015$; $\tilde{h}(4) = 0{\cdot}14595$; $\tilde{h}(5) = 0{\cdot}11524$.

1143. By the rectangle formula: $\tilde{h}(1) = 0{\cdot}49985$; $\tilde{h}(5) = 0{\cdot}19257$. Exact values: $\tilde{h}(1) = 0{\cdot}5$; $\tilde{h}(5) = 5/26 \approx 0{\cdot}19231$.

1146. (2) $T[\phi(t)] = -i\left[\phi(t) - 2\phi_a(t) - \frac{\phi_\infty + \phi_{-\infty}}{2}\right]$,

$$T[\psi(\bar{t})] = i\left[\psi(\bar{t}) - 2\psi_b(\bar{t}) - \frac{\psi_\infty + \psi_{-\infty}}{2}\right],$$

$$\phi_a(z) = \sum_{k=1}^{n} \frac{(-1)^{k-1} A_k}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}}\left(\coth\frac{z-a}{2}\right),$$

$$\psi_b(z) = \sum_{k=1}^{m} \frac{(-1)^{k-1} B_k}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}}\left(\coth\frac{z-b}{2}\right),$$

where $\sum_{k=1}^{n} \frac{A_k}{(z-a)^k}$, $\sum_{k=1}^{m} \frac{B_k}{(z-b)^k}$ are the principal values of the expansions of the functions $\phi(z)$ and $\psi(z)$ near the points a and b.

1147. (The notation is the same as in problem **1146**)

(1) $\phi(z) - i\ \mathrm{Im}\ \frac{\phi_\infty + \phi_{-\infty}}{2}$; (2) $\overline{\psi(\bar{z})} + i\ \mathrm{Im}\ \frac{\psi_\infty + \psi_{-\infty}}{2}$;

(3) $\phi(z) - \phi_a(z) - i\ \mathrm{Im}\ \frac{\phi_\infty + \phi_{-\infty}}{2} + \overline{\phi_a(\bar{z})}$;

(4) $\overline{\psi(\bar{z})} - \overline{\psi_b(\bar{z})} + i\ \mathrm{Im}\ \frac{\psi_\infty + \psi_{-\infty}}{2} + \psi_b(z)$;

(5) $\frac{1}{z-a} - \frac{1}{2}\coth\frac{z-a}{2} + \frac{1}{2}\coth\frac{z-\bar{a}}{2}$;

(6) $\coth(z-a) - \frac{1}{2}\coth\frac{z-a}{2} + \frac{1}{2}\coth\frac{z-\bar{a}}{2}$.

1148. $-\pi^2\tau^n$.

1149. $-\pi^2/\tau^n$.

1150. $-\pi^2\tau^n$.

1151. π^2/τ^n.

1152. $\mp\pi^2\tau^{p+q}$ (the sign $-$ if p and $p+q$ have the same sign, the sign $+$ if they have different signs; 0 is counted as the sign $+$).

1153. $-\dfrac{\pi^2}{\tau+bi}$.

1154. $-\pi^2 \cos \lambda\tau \cos \mu\tau$, if $\lambda > \mu$; $\pi^2 \sin \lambda\tau \sin \mu\tau$, if $\lambda < \mu$; $-\dfrac{\pi^2}{2} \cos 2\lambda\tau$, if $\lambda = \mu \neq 0$.

1155. $\mp i\pi^2 \dfrac{\sin \lambda\tau}{\tau+bi}$ (minus, if $b > 0$, plus, if $b < 0$).

1156. $\log \dfrac{1-\tau}{\tau} + \dfrac{\tau}{2}\left[\left(\log \dfrac{\tau}{1-\tau}\right)^2 - \pi^2\right]$.

1157. $\dfrac{1}{2} + \tau \log \dfrac{1-\tau}{\tau} + \dfrac{\tau^2}{2}\left[\left(\log \dfrac{\tau}{1-\tau}\right)^2 - \pi^2\right]$.

1158. $\dfrac{\tau^n}{2}\left[\left(\log \dfrac{\tau}{1-\tau}\right)^2 - \pi^2\right] + \sum_{k=1}^{n-1} \dfrac{1}{n-k}\left[\tau^k \log \dfrac{1-\tau}{\tau} + \left(\dfrac{1}{k} + \dfrac{\tau}{k-1} + \ldots + \dfrac{\tau^{k-1}}{1}\right)\right] - \sum_{p=0}^{n-1} \tau^{n-p-1}\left[\dfrac{1}{(p+1)^2} - \sum_{k=0}^{p} C_n^k \dfrac{(-1)^k}{(k+1)^2}\right]$.

1159. $\dfrac{1}{2}\left[\log^2 \dfrac{R+\tau}{R-\tau} - \pi^2\right]$.

1160. $\dfrac{\tau}{2}\left[\log^2 \dfrac{R+\tau}{R-\tau} - \pi^2\right] - 2R \log \dfrac{R+\tau}{R-\tau}$.

1161. $\dfrac{\tau}{2}\left[\log^2 \dfrac{R+\tau}{R-\tau} - \pi^2\right]$.

1162. $\dfrac{2R}{\tau}\left[\pi i - \log \dfrac{R+\tau}{R-\tau}\right] + \dfrac{1}{2}\left[\log^2 \dfrac{R+\tau}{R-\tau} - \pi^2\right]$.

1163. $\dfrac{1}{2}\left[\log^2 \dfrac{R+\tau}{R-\pi} - \pi^2\right] + 2\pi Ri + \tau \log \dfrac{R+\tau}{R-\tau}$.

1164. $\dfrac{1}{2\tau^2} \log^2 \dfrac{R+\tau}{R-\tau} - \dfrac{2\pi i}{\tau}$.

CHAPTER IX

1175. $f(z) = \sum_{n=0}^{\infty} \frac{(z-a)^n}{(1-a)^{n+1}}$; this expansion continues $f(z)$ analytically inside the circle $|z-a| < |1-a|$ which does not lie entirely within the circle $|z| < 1$, if a does not belong to the interval $[0,1)$.

1176. $f(z) = \log \frac{2}{3} + \sum_{n=1}^{\infty} \frac{\left(\frac{2}{3}\right)^n \left(z+\frac{1}{2}\right)^n}{n}$; the circle of convergence of this series is $\left|z+\frac{1}{2}\right| < \frac{3}{2}$.

1191. SOLUTION. The substitution $e^t = x$ reduces the integral to the form $f(s) = \int_1^{\infty} \frac{\sin x \, dx}{x^s}$ $(x^s = e^{s \log x})$. Integrating by parts, we obtain $f(s)$ $= \cos 1 - \int_1^{\infty} \frac{\cos x}{x^{s+1}} dx$. The latter integral converges in the half-plane Re $s > -1$.

1194. $0 < \operatorname{Re} z < 1$, $-1 < \operatorname{Re} z < 1$.

1196. The point $z = 1$ is a simple pole with residue one.

1197. SOLUTION. Let us use the notation $f_1(z) = \int_0^{\infty} e^{-t} \phi(zt) dt$. The analyticity of the function $f_1(z)$ in the circle $|z| < 1$ follows from the result of problem **535** and the general properties of Laplace integrals (see page 110). Integrating by parts $(n+1)$ times and using the inequalities of problem **535**, we obtain for $|z| < 1$

$$f_1(z) = -\sum_{k=0}^{n} z^k \left[e^{-t} \phi^{(k)}(zt)\right]_0^{\infty} + z^{n+1} \int_0^{\infty} e^{-t} \phi^{(n+1)}(zt) dt$$

$$= \sum_{k=0}^{n} a_n z^n + z^{n+1} \int_0^{\infty} e^{-t} \phi^{(n+1)}(zt) dt.$$

From the estimate for $|\phi^{(n+1)}|$ it follows that the second term on the right hand side of the last equality tends to zero as $n \to \infty$ $(|z| < r)$. In order to prove the second assertion let us take any point $z \in G$. Then, as is not difficult to prove, inside and on the boundary of the circle on Oz as diameter there are no singularities of the function $f(z)$. Hence for a sufficiently small $\delta > 0$ within and on the boundary C of a circle of radius $(|z|/2)+\delta$, concentric with the one already constructed, the function $f(z)$ is also analytic. Thus

the equality $c_n = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta^{n+1}}d\zeta$ holds for the coefficients c_n of the expansion $f(z) = \sum_{n=0}^{\infty} c_n z^n$ and consequently,

$$\phi(zt) = \frac{1}{2\pi i}\sum_{n=0}^{\infty}\frac{z^n t^n}{n!}\int_C \frac{f(\zeta)dz}{\zeta^{n+1}}.$$

Since the series $\sum_{n=0}^{\infty}\frac{z^n t^n f(\zeta)}{n!\zeta^{n+1}}$ converges uniformly on C, it follows that $\phi(zt) = \frac{1}{2\pi i}\int_C f(\zeta)e^{zt/\zeta}\frac{d\zeta}{\zeta}$. The maximum of the quantity Re (z/ζ) on C is equal to $\frac{|z|}{|z|+\delta} = q < 1$ (for the proof of this assertion it is sufficient to consider the case when z is real and positive as a rotation round the coordinate origin does not change the quantity Re (z/ζ)) and consequently, $|\phi(zt)| < Ae^{qt}$ (A is a constant). It follows from this that the integral $\int_0^{\infty} e^{-t}\phi(zt)dt$ converges.

1201. If $\mathrm{Log}^{(1)}\frac{z-z_2}{z-z_1}$ is the branch of the function $\mathrm{Log}\frac{z-z_2}{z-z_1}$, which is analytic in the z-plane with a cut along the arc γ_1 and which tends to 0 at $z = \infty$, then $F^-(z) = (a-b)\,\mathrm{Log}^{(1)}\frac{z-z_2}{z-z_1}$, $F^+(z) = (a-b)\,\mathrm{Log}^{(1)}\frac{z-z_2}{z-z_1} + 2\pi bi$; within G^+: $F^-(z) = (a-b)\,\mathrm{Log}^{(1)}\frac{z-z_2}{z-z_1} - 2\pi i(a-b)$ for analytic continuation through γ_1 and $F^-(z) = (a-b)\,\mathrm{Log}^{(1)}\frac{z-z_2}{z-z_1}$ for analytic continuation through γ_2.

1203. (1) $z = \pm 3$; (2) $z = \pm 2i$; (3) $z = \pm 2i$. In all three cases the values of $w'(z)$ are different.

1205. Above $z = 1$ there are two elements: $z = 1+t$,

$$w_1 = \frac{1-(1-t)^{\frac{1}{2}}}{t} = \frac{1}{2} + 8t + \ldots, \quad |t| < 1,$$

and

$$w_2 = \frac{1+(1-t)^{\frac{1}{2}}}{t} = \frac{2}{t} - \frac{1}{2} - \frac{1}{8}t + \ldots, \quad 0 < |t| < 1;$$

above $z = 2$ there is one algebraic element: $z = 2+t^2$, $w = \frac{1}{1+it} = 1 - it - t^2 + \ldots$, $|t| < 1$.

1206. Above $z = 1$ there are two algebraic elements: $z = 1+t^2$, $w = \pm i\sqrt{2}\left(1-\frac{t}{2}\right)^{\frac{1}{2}} = \pm i\sqrt{2}\left(1-\frac{t}{4}+\ldots\right)$, $|t| < 2$; above $z = 5$ there is one algebraic element: $z = 5+t^2$, $w = \frac{t}{2}\left(1-\frac{1}{4}t^4+\ldots\right)$, $|t| < 2$, and two regular elements: $z = 5+t$, $w = \pm 2i\left(1+\frac{t}{32}+\ldots\right)$, $|t| < 4$; above $z = \infty$ there is one algebraic element: $z = t^{-4}$, $w = \frac{1}{t}-t+\ldots$, $0 < |t| < \sqrt[4]{(1/5)}$.

1207. Above $z = 1$ there are two algebraic elements: $z = 1+t^2$, $w = \pm (1+t)^{\frac{1}{2}} = \pm\left(1+\frac{1}{2}t-\ldots\right)$, $|t| < 1$.

1208. Above $z = \infty$ there are two algebraic elements: $z = t^{-2}$, $w = \pm\frac{1}{t}(1-at)^{\frac{1}{2}}(1-bt)^{\frac{1}{2}} = \pm\frac{1}{t}\left(1-\frac{a+b}{2}t+\ldots\right)$, $0 < |t| < \min\left(\frac{1}{|a|}, \frac{1}{|b|}\right)$.

1209. $z = 0$ is an essential singularity, two-valued: $z = t^2$, $w = e^{\frac{1}{t}} = 1+\frac{1}{t}+\frac{1}{2!t^2}+\ldots$, $0 < |t| < \infty$.

1210. Above $z = 0$ there is an algebraic element: $z = t^2$, $w = \frac{\sin t}{t^4} = \frac{1}{t^3}-\frac{1}{3!t}+\frac{1}{5!}t-\ldots$, $0 < |t| < \infty$.

1211. Above $z = 0$ there is the algebraic element: $z = t^3$, $w = \cot t = \frac{1}{t}-\frac{t}{3}+\ldots$, $0 < |t| < \pi$.

1212. Above $z = 0$ there is the algebraic element: $z = t^2$, $w = t\left(1-\frac{t^4}{12}+\ldots\right)$, $|t| < \sqrt{\pi}$.

1213. $w = 1/4$, $w = \infty$ are algebraic branch points of the first order.

1214. $w = \pm 2$ are algebraic branch points of the first order; $w = \infty$ is an algebraic branch point of the second order.

1215. $w = 1/4$, $w = \infty$ are algebraic branch points of the first order.

1216. $w = 0$, $w = \infty$ are algebraic branch points of the first order.

1217. $w = \frac{1}{2}e^{\pm i\alpha}$ ($a = \cos\alpha$) are algebraic branch points of the first order.

1218. $w = \infty$ is an algebraic branch point of the $(n-1)$th order. To every zero of the derivative $w'(z)$ of order k there corresponds an algebraic branch point of the function $z(w)$ of the same order.

1219. To the zeros of the derivative $w'(z)$ there correspond algebraic branch points just as in the preceding case. To the poles of the function $w(z)$ of order greater than one there correspond algebraic branch points the order of which

is one less than the order of the pole. If at ∞ the function $w(z) = w_0 + \frac{c_{-k}}{z^k} + \ldots$ ($k > 1$), then $z = \infty$ corresponds to an algebraic branch point of order $(k-1)$ above $w = w_0$.

1220. The surface for $z(w)$ is the same as for $\sqrt[n]{w}$; its branch points are situated above $w = 0$, and $w = \infty$ and correspond to $z = -n$, $z = \infty$. To sheets of the w-plane with the cuts $0 < u < \infty$, $v = 0$ there correspond angles $2\pi/n$ with vertices at the point $z = -n$. As $n \to \infty$ these angles are transformed into horizontal strips of width 2π, the function $w(z)$ into e^z, and the surface for $z(w)$ into the surface for $\operatorname{Log} w$.

1221. The surface for $z(w)$ is the same as for $\sqrt[n]{w}$. To the sheets of the w-plane with the cuts $-\infty < u < 0$, $v = 0$ there correspond figures formed from two circular segments with angles $2\pi/n$ at the points $z = a$, $z = b$ (Fig. 73, where $n = 3$)†.

1222. The surface for $z(w)$ is obtained by joining together two sheets of the w-plane with the cuts $|u| < 1$, $v = 0$, corresponding to the domains $|z| < 1$ and $|z| > 1$. The branch points are situated above $w = \pm 1$ and correspond to $z = \pm 1$. To the polar net $|z| = r$, $\arg z = \phi$ there correspond the ellipses and hyperbolas with foci ± 1:

$$\frac{x^2}{\left[\frac{1}{2}\left(r+\frac{1}{r}\right)\right]^2} + \frac{y^2}{\left[\frac{1}{2}\left(r-\frac{1}{r}\right)\right]^2} = 1; \quad \frac{x^2}{\cos^2\phi} - \frac{y^2}{\sin^2\phi} = 1.$$

1223. $w(z) = z/(1-z)^2 = z + 2z^2 + 3z^3 + \ldots$ is a well-known extremal function in the theory of single-valued conformal mappings (see[4, Chapter XIII, § 1]). It maps the unit disk $|z| < 1$ onto the w-plane with the cut $-\infty < u < -1/4$, $v = 0$. The surface $z(w)$ is obtained by joining together two such sheets.

1224. The surface for $z(w)$ is obtained by the sequential attachment of $2n$ sheets of the w-plane with the cuts $1 < |u| < \infty$, $v = 0$. It has $2n$ branch points of the first order above $w = \pm 1$ and 2 branch points of the $(n-1)$th order above $w = \infty$. The basic mapping is shown in Fig. 74. The function $w(z)$ is automorphic (invariant) with respect to groups of linear transformations, generated by the transformations $T = \omega z$ ($\omega = e^{\frac{2\pi i}{n}}$), $S = 1/z$. These transformations correspond to transformations of the surface for $z(w)$ into itself during which the sheets are cyclically permuted and the projections of points onto the w-plane are preserved.

1225. The surface for $z(w)$ is $2n$-sheeted with a branch point of the $(2n-1)$th order above $w = 0$, corresponding to $z = \infty$, and with $2n$ branch points of the first order of which n are situated above $w = \infty$ and correspond to the points $z_k = e^{\frac{\pi i}{n}} \omega^k$ ($\omega = e^{\frac{2\pi i}{n}}$, $k = 0, 1, 2, \ldots, n-1$), and n above the

† In the figures to the problems of this chapter corresponding points are denoted by identical letters. As a rule, the symbols Ω and Ω' are used for the points at infinity.

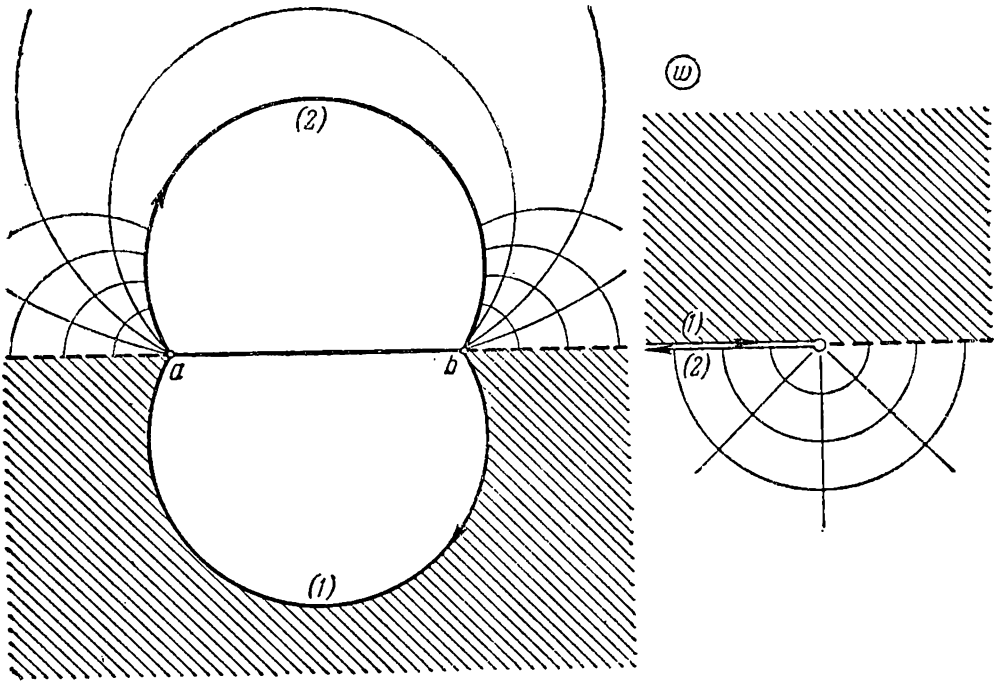

FIG. 73

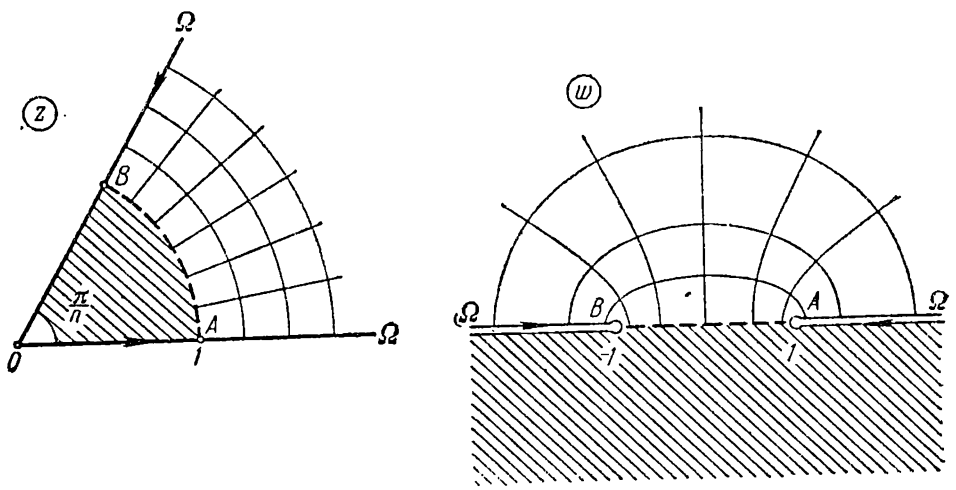

FIG. 74

points $w_k' = \left(\frac{2n-1}{2n}\right)^2 z_k'$ corresponding to the points $z_k' = \sqrt[n]{(1/(2n-1))}\,\omega^k$. The basic mapping is shown in Fig. 75. The rest is obtained by continuation by the symmetry principle.

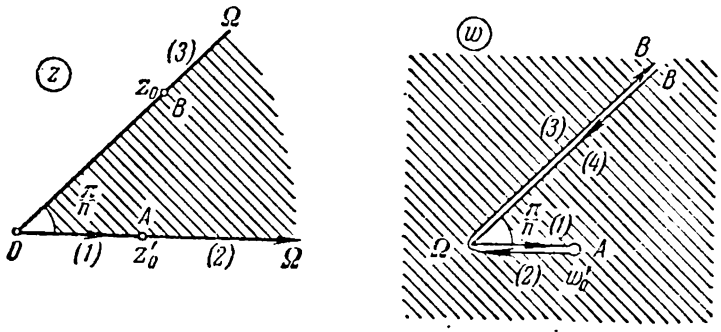

FIG. 75

1226. The surface for $z(w)$ has n sheets with a branch point of the $(n-1)$-th order above $w = \infty$, corresponding to $z = \infty$, and $n - 1$ branch points of the first order, situated above $w_k = z_k\left(1-\frac{1}{n}\right)$ and corresponding to $z_k = \omega^k$ ($\omega = e^{\frac{2\pi i}{n-1}}$, $k = 0, 1, \ldots, n-2$). In order to construct the surface it is necessary to attach to its zero sheet (the w-plane with $n - 1$ radial cuts originating from the w_k) along each cut criss cross one sheet (the w-plane with one radial cut). The basic mapping is shown in Fig. 76. To the disk $|z| < 1$ there corresponds the interior of an epicycloid (for $n = 2$, a cardiod).

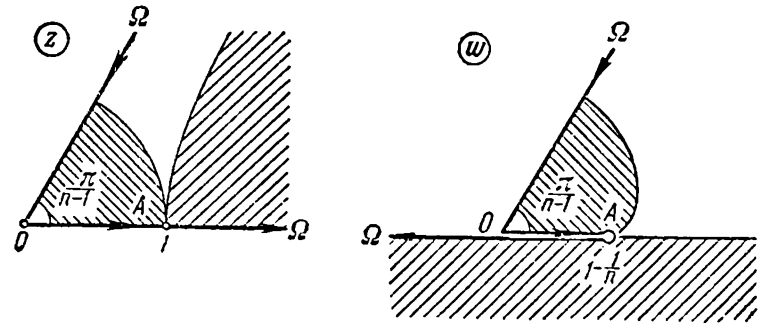

FIG. 76

1227. The surface for $z(w)$ has $(n+1)$ sheets with a branch point of the $(n - 1)$th order above $w = \infty$, corresponding to $z = \infty$, and $n+1$ branch

points of the first order situated above $w_k = \overline{z}_k\left(1+\frac{1}{n}\right)$ and corresponding to $z_k = \omega^k$ ($\omega = e^{\frac{2\pi i}{n+1}}$, $k = 0, 1, \ldots, n$). The basic mapping is shown in Fig. 77 The domain $|z| < 1$ corresponds to the exterior of a hypocycloid (for $n = 1$ a segment).

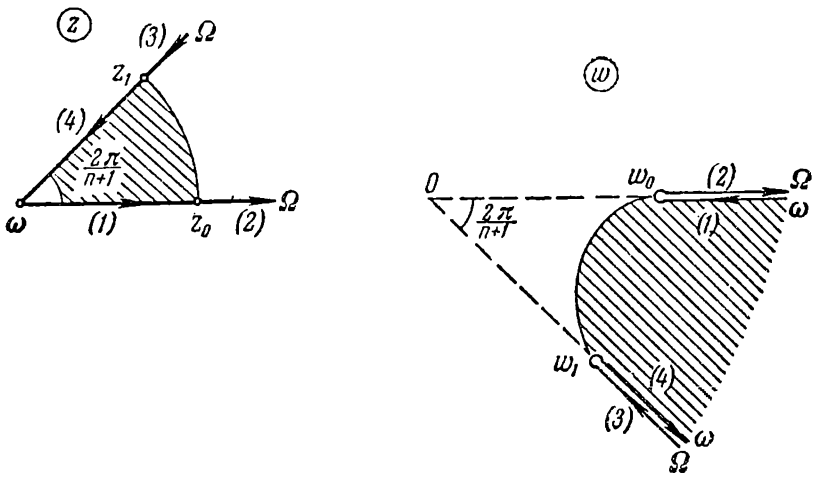

FIG. 77

1228. The surface for $z(w)$ has n-sheets with a branch point of the $(n-1)$th order above $w = \infty$ corresponding to $z = \infty$, and $n-1$ branch points of the first order situated above $w_k = \frac{(-1)^k}{2^{n-1}}$ and corresponding to $z_k = \cos\frac{k\pi}{n}$ ($k = 1, 2, \ldots, n-1$). To construct the surface it is neeessary to attach to its zero sheet (the w-plane with the cut $-\infty < u \leqslant -\frac{1}{2^{n-1}}$, $v = 0$) sequentially $n-2$ sheets with the two cuts $\frac{1}{2^{n-1}} \leqslant |u| < \infty$, $v = 0$, then to the last of them to attach another sheet having the cut $-\infty < u \leqslant -\frac{1}{2^{n-1}}$, $v = 0$, if n is an even number, and the cut $\frac{1}{2^{n-1}} \leqslant u < \infty$, $v = 0$ if n is odd. In the mapping $w(z)$ ellipses and hyperbolas with foci ± 1 pass into ellipses and hyperbolas with foci $\pm\frac{1}{2^{n-1}}$. The change of the semi-axes is obtained from the relations $z = \text{const}$, $w = \text{const}$. Fig. 78 shows the dissection of the z-plane into domains corresponding to the half-planes $v > 0$ and $v < 0$ (the former are shaded) for $n = 5$.

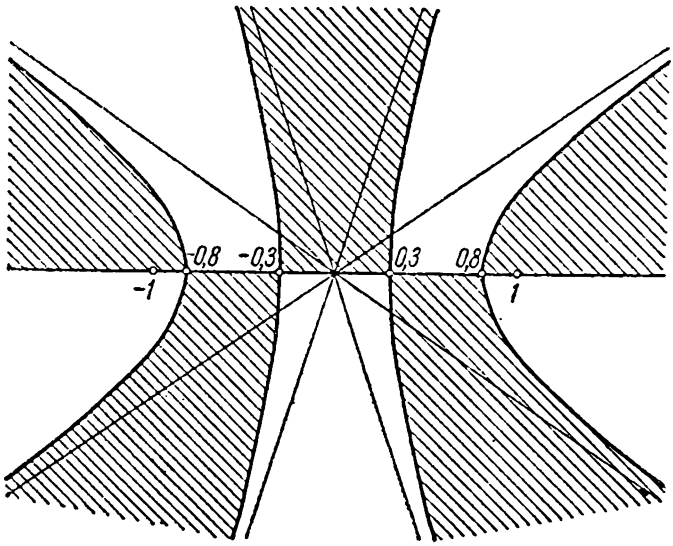

FIG. 78

1229. $w = 0$, $w = \infty$ are logarithmic branch points, $w = 1$ is a simple pole for one of the branches of the function $z(w)$.

1230. The function $z(w)$ has one algebraic branch point of the first order above each of the points $w = e^{\pm 2i}$ and two logarithmic branch points above each of the points $w = 0$ and $w = \infty$. Its Riemann surface is obtained by joining together two specimens of the Riemann surface of the function Log w along cuts made on them above the arc connecting on the w-plane the points $w = e^{\pm 2i}$.

1231. The function $z(w)$ has above $w = \pm\sqrt{2}$ an infinite set of algebraic branch points of the first order and above $w = \infty$ two logarithmic branch points.

1232. The function $z(w)$ has one algebraic branch point of the first order above each of the points $w_k = \dfrac{\sin z_k}{z_k}$ (the z_k are the roots of the equation $\tan z = z$; they are all real), two logarithmic branch points above $w = \infty$ and an indirectly critical singularity above $w = 0$, the limit of the given algebraic branch points (see R. NEVANLINNA, *Eindeutige analytische Funktionen*, page 238, Berlin, 1936).

1233. $z = \cos^{-1} w = \dfrac{1}{i}\,\mathrm{Log}\,[w+\sqrt{(w^2-1)}]$. The surface for $z(w)$ has an infinite number of sheets with two logarithmic branch points above $w = \infty$ and algebraic branch points of the first order above $w = \pm 1$ corresponding to $z = k\pi$ ($k = 0, \pm 1, \pm 2, \ldots$); obtained by joining together an infinite number of w-planes with the cuts $1 \leqslant |u| < \infty$, $v = 0$, which correspond to the vertical strips $k\pi < x < (k+1)\pi$ (Fig. 79).

1234. $z = \sin^{-1} w = \pi/2 - \cos^{-1} w$. The surface for $z(w)$ is the same as for $\cos^{-1} w$ (Fig. 80).

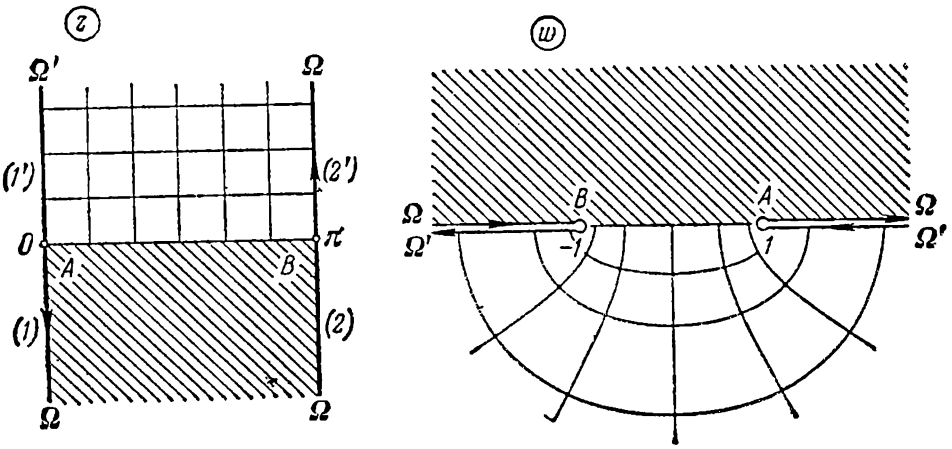

FIG. 79

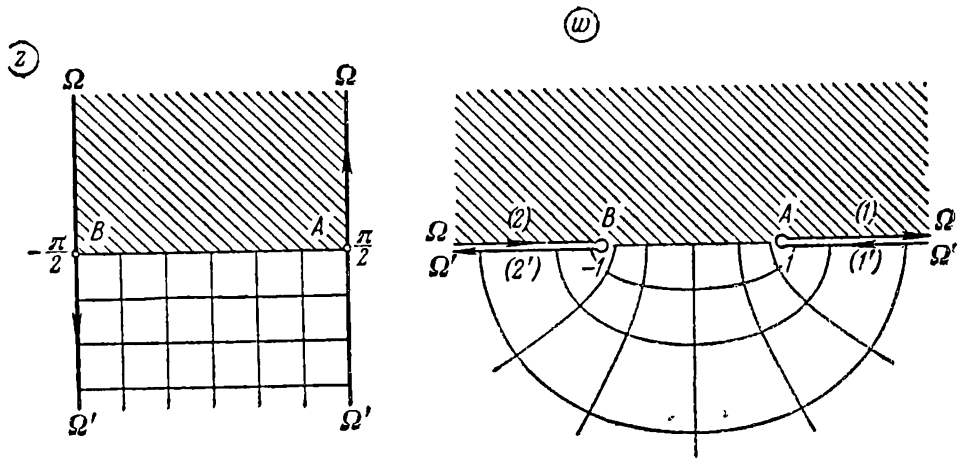

FIG. 80

1235. $z = \tan^{-1} w = \dfrac{1}{2i} \operatorname{Log} \dfrac{i-w}{i+w}$. The surface for $z(w)$ has an infinite number of sheets and two logarithmic branch points above $w = \pm i$; it is obtained by joining together an infinite number of w-planes with the cut $u = 0$, $|v| \leqslant 1$, corresponding to the vertical strips $k\pi < x < (k+1)\pi$ (Fig. 81).

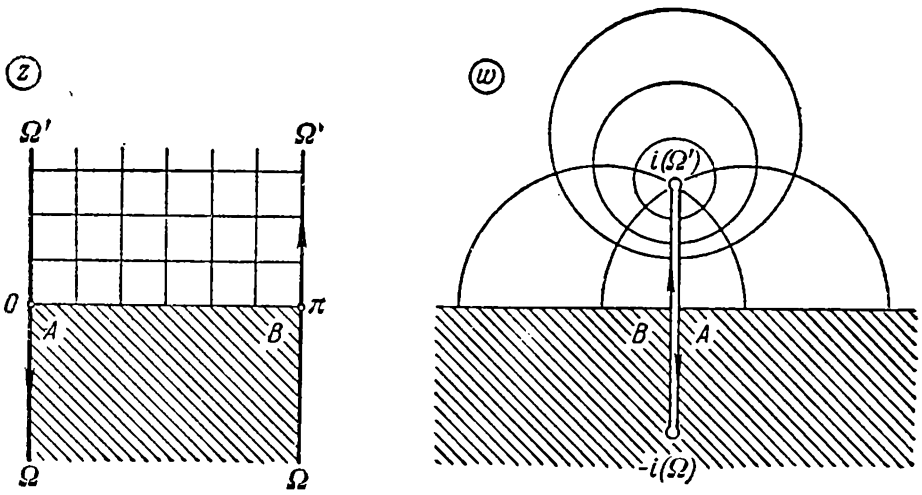

FIG. 81

1236. $z = \cot^{-1} w = \pi/2 - \tan^{-1} w$. The surface for $z(w)$ is the same as for $\operatorname{Tan}^{-1} w$ (Fig. 82).

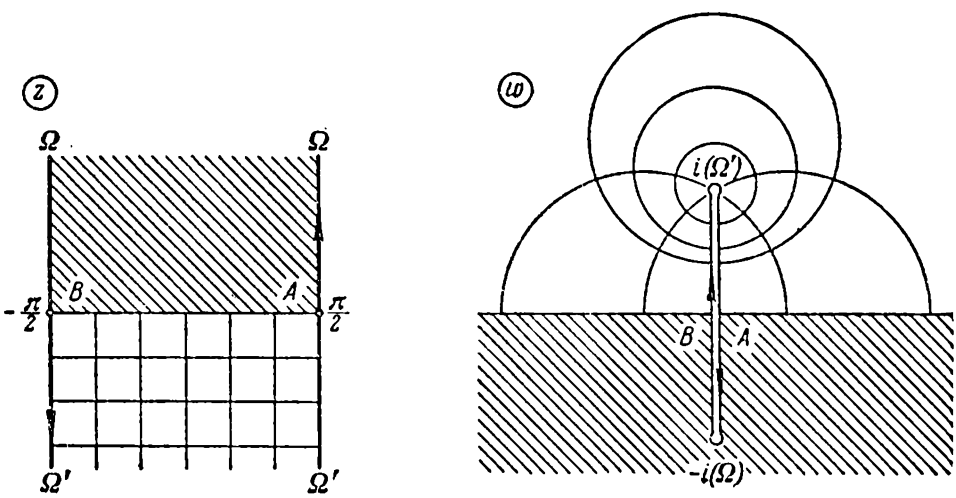

FIG. 82

1237. $z = \cosh^{-1} w = \operatorname{Log}(w+\sqrt{(w^2-1)})$; $\cosh z = \cos iz$. The surface for $z(w)$ is the same as that for $\cos^{-1} w$ (Fig. 83).

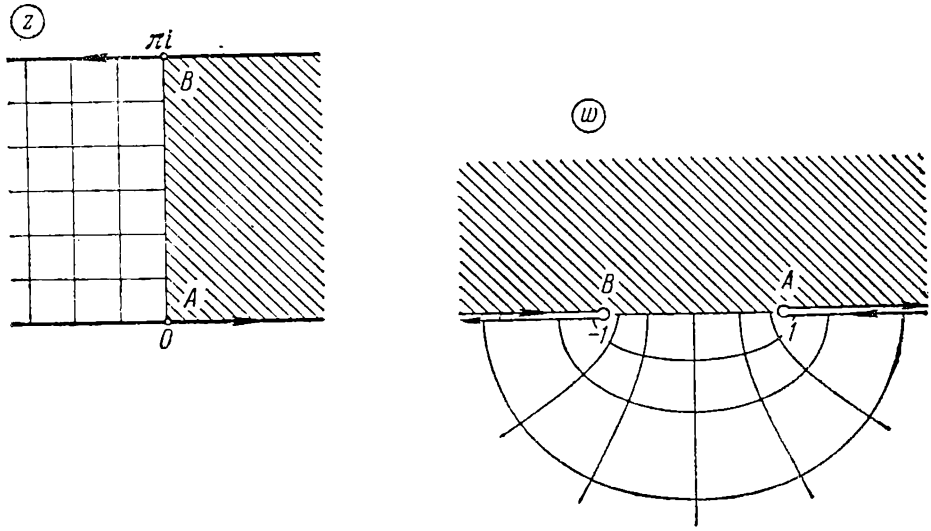

Fig. 83

1238. $z = \sinh^{-1} w = \operatorname{Log}(w+\sqrt{(w^2+1)})$; $\sinh z = -i \sin iz$. The surface for $z(w)$ is obtained from the preceding one by a rotation through $\pi/2$ about the coordinate origin (Fig. 84).

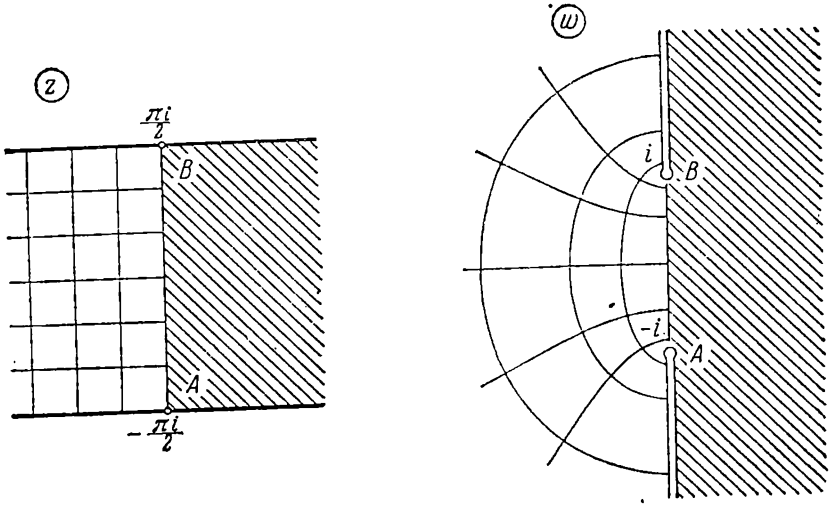

Fig. 84

1239. $z = \tanh^{-1} w = \frac{1}{2} \operatorname{Log} \frac{1+w}{1-w}$; $\tanh z = -i \tan iz$. The surface for $z(w)$ is obtained from the surface for $\tan^{-1} w$ by a rotation through $\pi/2$ about the coordinate origin (Fig. 85).

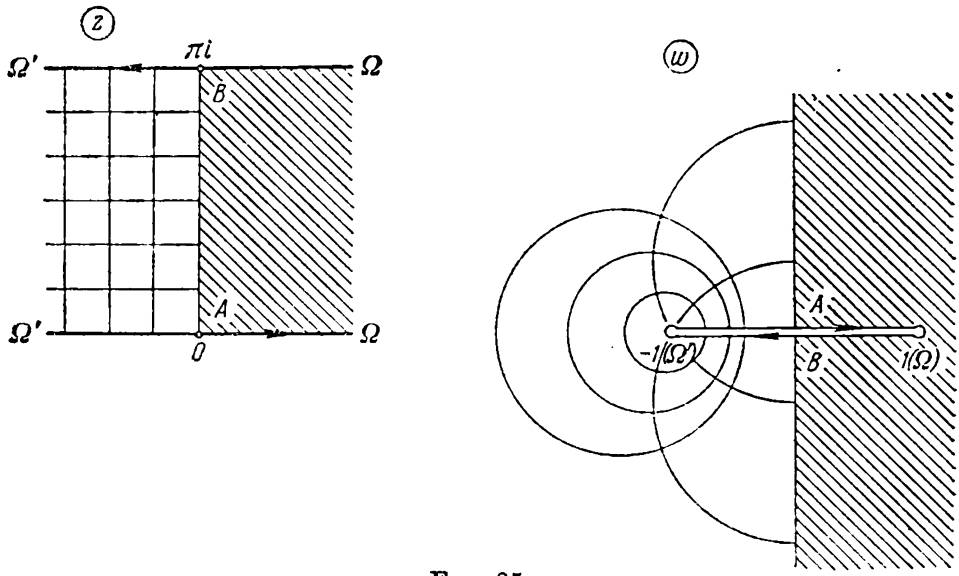

FIG. 85

1240. $z = \coth^{-1} w = \frac{1}{2} \operatorname{Log} \frac{w+1}{w-1}$; $\coth z = i \cot iz$. The surface for $z(w)$ is the same as that for $\tanh^{-1} w$ (Fig. 86).

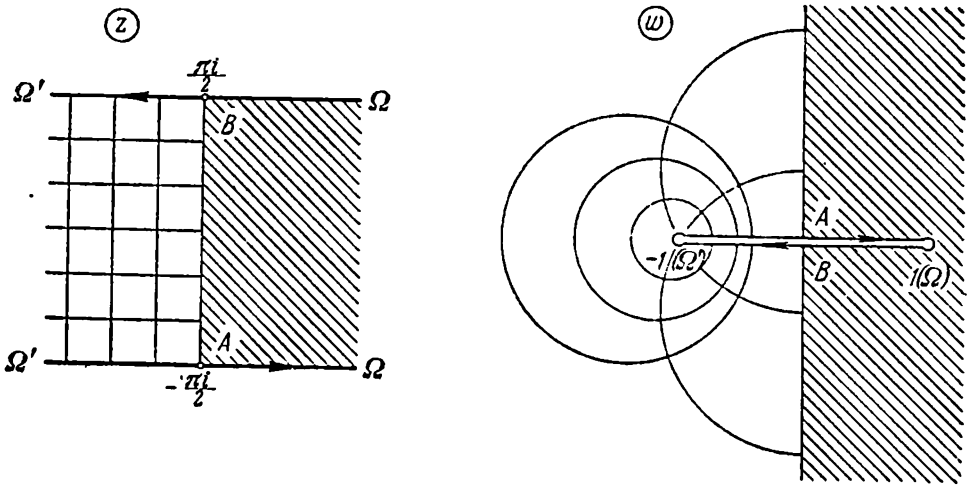

FIG. 86

1241. The surface for $z(w)$ is constructed as follows: we make on the w-plane the horizontal cuts $-\infty < u \leqslant 1$, $v = (2k+1)\pi$, $(k = 0, \pm 1, \pm 2, \ldots)$ and along each of these we attach one sample of the w-plane with the same (single) cut. The surface constructed for $z(w)$ is based on the fact that $w(z)$ maps each strip $2k\pi < y < (2k+1)\pi$ onto the strip $2k\pi < v < (2k+2)\pi$ carrying the w-plane joined along the cut $-1 \leqslant u < \infty$, $v = (2k+1)\pi$ (see Fig. 87; the sign + denotes that the domains have to be joined together).

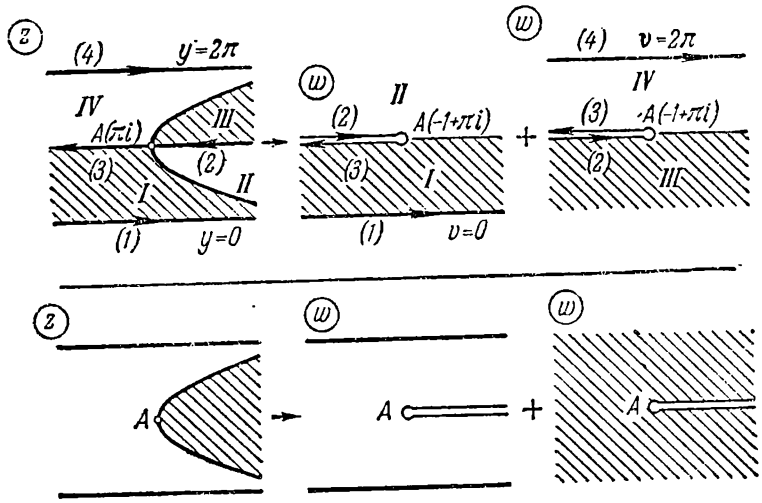

FIG. 87

1242. (1) Let F be the Riemann surfaee onto which the function $w = R(\zeta)$ maps the ζ-plane. In order to construet the Riemann surface of the function $z(w)$ it is necessary to join together infinitely many specimens of the surface F with a cut along the arc connecting on F the points $w(0)$ and $w(\infty)$ (similar to the construction of the Riemann surface of the function Log w). The resulting Riemann surface has two logarithmic branch points at the ends of the arc of attachment and an infinite number of algebraie branch points belonging to the surfaces F.

(2) To construct the Riemann surface of the function $z(w)$ we join together an infinite number of specimens of the surface F alternately along the cuts going from the points $w(\pm 1)$ to the point $w(\infty)$ (similar to the construction of the Riemann surface of the function $\sin^{-1} w$). The resulting Riemann surfaee has two logarithmic branch points above $w(\infty)$ and in addition to the algebraic branch points of part (1) it has also an infinite number of algebraic branch points of the first order at the points $w(\pm 1)$ (if $w(+1)$ ór $w(-1)$ is an algebraic branch point of order k of the surfaee F, then for $z(w)$ it will be an algebraic branch point of order $2k+1$). The investigation of the function $z(w)$ can also be reduced to the preceding case by the substitution $z_1 = iz$.

1243. All the Riemann surfaces of $w(z)$ have two sheets and have algebraic branch points of the first order above the points: (1) $z = a$, $z = b$; (2) $z = a$, $z = b$, $z = c$, $z = \infty$; (3) $z = a_k$ and $z = \infty$, if n is odd. For the construction of the surfaces we take two sheets of the z-plane with cuts going from the points indieated above to ∞, and join them along identical cuts.

1244. All the Riemann surfaces for $w(z)$ have three sheets and have algebraic branch points of the second order above the points: (1) $z = a$, $z = \infty$; (2) $z = a$, $z = b$, $z = \infty$; (3) $z = a$, $z = b$, $z = c$; (4) $z = a_k$ and $z = \infty$, if n is not a multiple of 3. For the construction of the surfaces we take three sheets of the z-plane with cuts going from the given points to ∞, on which we define the three single-valued branches w, ωw, $\omega^2 w$ ($\omega = e^{\frac{2\pi i}{3}}$). On going round the branch points $w(z)$ acquires a factor ω on account of one of the factors of the root, hence the order of the attachment of the cuts is the same, cyclic, along all the cuts (see the sketch in Fig. 88).

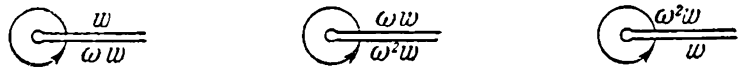

FIG. 88

1245. The Riemann surface for $w(z)$ has n sheets with algebraic branch points of the $(n-1)$th order above $z = a$, $z = b$, $z = c$, $z = \infty$. The surface is obtained by attaching n sheets of the z-plane with cuts going from the points $z = a$, $z = b$, $z = c$ to ∞. The attachment is cyclic, simultaneous on all three cuts. The sheets correspond to the single-valued branches of the function $\omega^k w$ ($\omega = e^{2\pi i/n}$, $k = 0, 1, \ldots, n-1$).

1246. The Riemann surface of $w(z)$ has six sheets with an algebraic branch point of the fifth order above $z = \infty$, two algebraic branch points of the second order above each of the points $z = a$, $z = b$ and three algebraic branch points of the first order above $z = c$. The surface is obtained by joining together six sheets of the z-plane with a cut along a curve going from a to b, from b to c and from c to ∞. These sheets correspond to the single-valued branches: w_1+w_2, ωw_1+w_2, $\omega^2 w_1+w_2$, w_1-w_2, ωw_1-w_2, $\omega^2 w_1-w_2$, where $\omega = e^{2\pi i/3}$ and w_1, w_2 are the single-valued branches $\sqrt[3]{[(z-a)/(z-b)^2]}$ and $\sqrt{(z-c)}$. The detour round a cyclically connects sheets 1, 2, 3 and 4, 5, 6, that round b sheets 1, 3, 2 and 4, 6, 5 (connected twice in half-sheets), that round c the half-sheets 1, 2, 4, 5 (shown in Fig. 89); 2, 3, 5, 6; 3, 1, 6, 4; that round ∞ cyclically the sheets 1, 6, 2; 4, 3, 5 (see the diagrams in Fig. 89).

1247. The Riemann surface for $w(z)$ has six sheets with two algebraic branch points of the second order above $z = 0$, one algebraic branch point of the first order above $z = 1$ and an algebraic branch point of the fifth order above $z = \infty$. In order to construct it two samples of the surface of the function $\sqrt[3]{z}$, have to be joined together, each of them having on one of the sheets a cut along the ray $y = 0$, $1 \leqslant x < \infty$.

1248. The Riemann surface of $w(z)$ has two sheets with algebraic branch points of the first order above the points $z = k\pi$ ($k = 0, \pm 1, \pm 2, \ldots$); above $z = \infty$ the surface has a transcendental singularity the limit of algebraic branch points. In order to construct the surface take two sheets of the z-plane with cuts going from the algebraic branch points to ∞ (for example, along rays parallel to the imaginary axis) and join the sheets together along identical cuts.

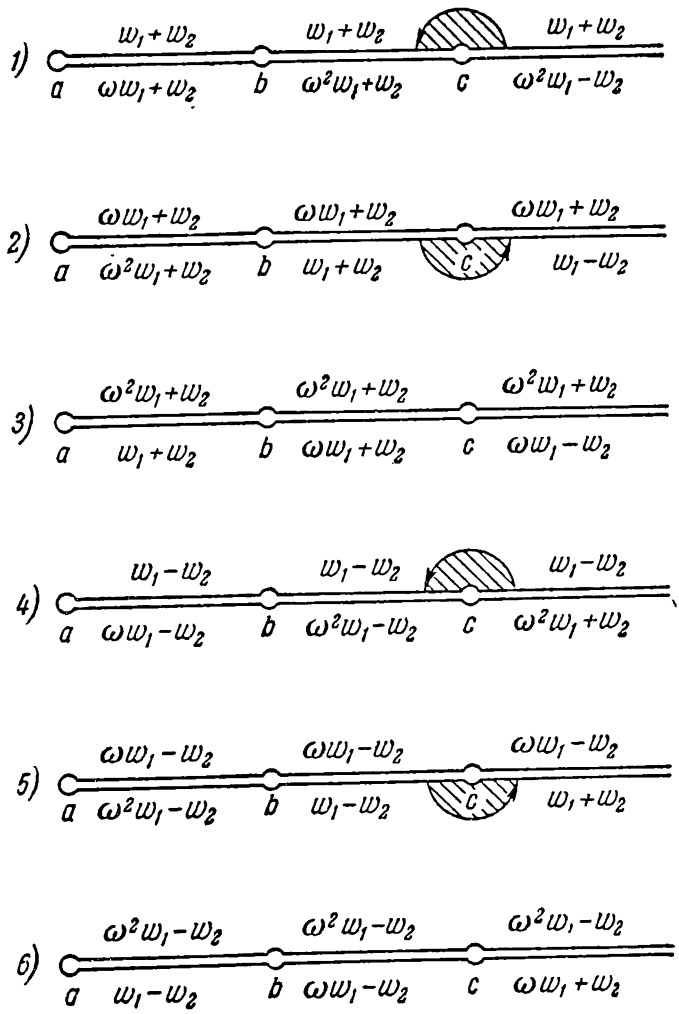
1)
w_1+w_2 w_1+w_2 w_1+w_2
a ωw_1+w_2 b $\omega^2 w_1+w_2$ c $\omega^2 w_1-w_2$
2)
ωw_1+w_2 ωw_1+w_2 ωw_1+w_2
a $\omega^2 w_1+w_2$ b w_1+w_2 c w_1-w_2
3)
$\omega^2 w_1+w_2$ $\omega^2 w_1+w_2$ $\omega^2 w_1+w_2$
a w_1+w_2 b ωw_1+w_2 c ωw_1-w_2
4)
w_1-w_2 w_1-w_2 w_1-w_2
a ωw_1-w_2 b $\omega^2 w_1-w_2$ c $\omega^2 w_1+w_2$
5)
ωw_1-w_2 ωw_1-w_2 ωw_1-w_2
a $\omega^2 w_1-w_2$ b w_1-w_2 c w_1+w_2
6)
$\omega^2 w_1-w_2$ $\omega^2 w_1-w_2$ $\omega^2 w_1-w_2$
a w_1-w_2 b ωw_1-w_2 c ωw_1+w_2

FIG. 89

1249. † F_z and F_w are each obtained by joining together two planes with cuts $[-1, 1]$ (Fig. 90).

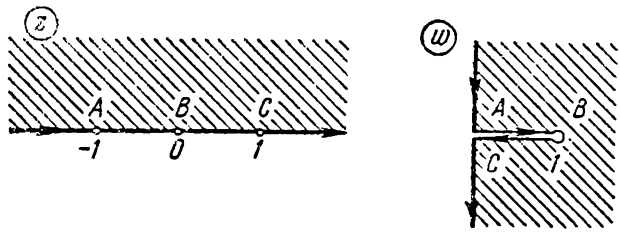

FIG. 90

1250. F_z is obtained by joining together two z-planes with the cuts $(-\infty < x \leqslant 0, y = 0)$, F_w two w-planes with the cuts $\left[-\frac{i}{2}, \frac{i}{2}\right]$ (Fig. 91).

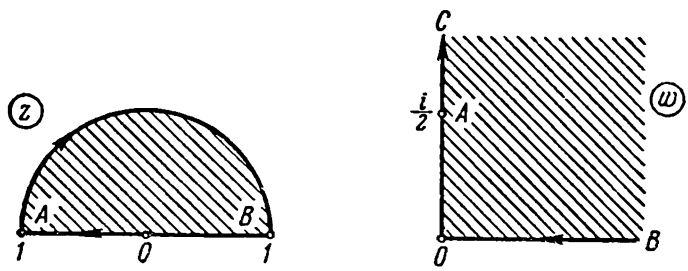

FIG. 91

1251. F_z and F_w are each obtained by joining together three planes having in cyclic order cuts along two of the segments $[0, \sqrt[3]{4}]$, $[0, \omega\sqrt[3]{4}]$, $[0, \omega^2\sqrt[3]{4}]$, where $\omega = e^{2\pi i/3}$ (Fig. 92).

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† In the answers to problems **1249–1254** by F_w are denoted the surfaces for $z(w)$, that is, surfaces above the w-plane and by F_z the surfaces of the function $w(z)$ above the z-plane.

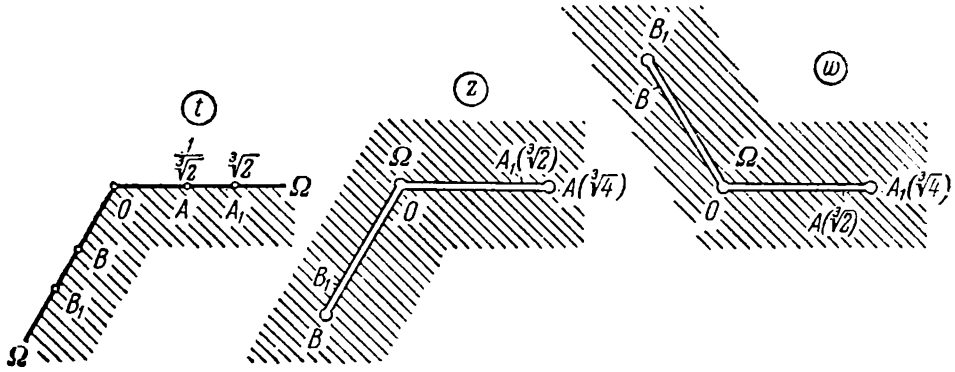

FIG. 92

1252. F_z is obtained by joining together two z-planes with the cuts $\left[0, \frac{3}{2}\right]$, F_w by joining to the w-plane with the cuts γ_1: $\left[i\frac{3\sqrt{3}}{2}, +i\infty\right]$, γ_2: $\left[-i\frac{3\sqrt{3}}{2}, -i\infty\right]$ two other w-planes with the cuts γ_1 and γ_2 respectively (Fig. 93).

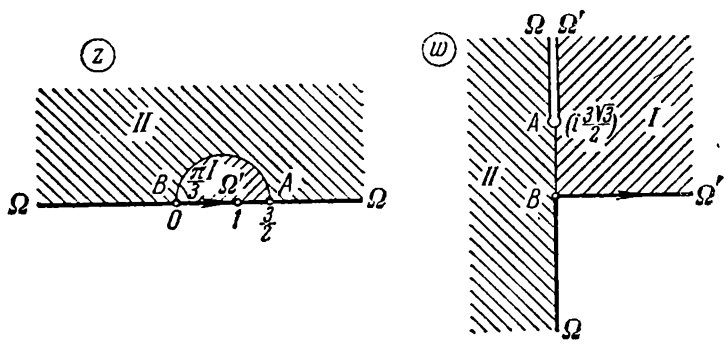

FIG. 93

1253. F_z is obtained by joining together two z-planes with cuts $[-1, 1]$, F_w by joining together three w-planes having respectively one cut $[-ib, ib]$, two cuts $[-a, a]$, $[0, ib]$ and two cuts $[-a, a]$, $[0, -ib]$, where $a = \frac{\sqrt{5}-1}{2}\sqrt{(\sqrt{5}-2)}$, $b = \frac{\sqrt{5}+1}{2}\sqrt{(\sqrt{5}+2)}$. The mapping and the three w-planes with cuts are shown in Fig. 94 (the figures are not to scale).

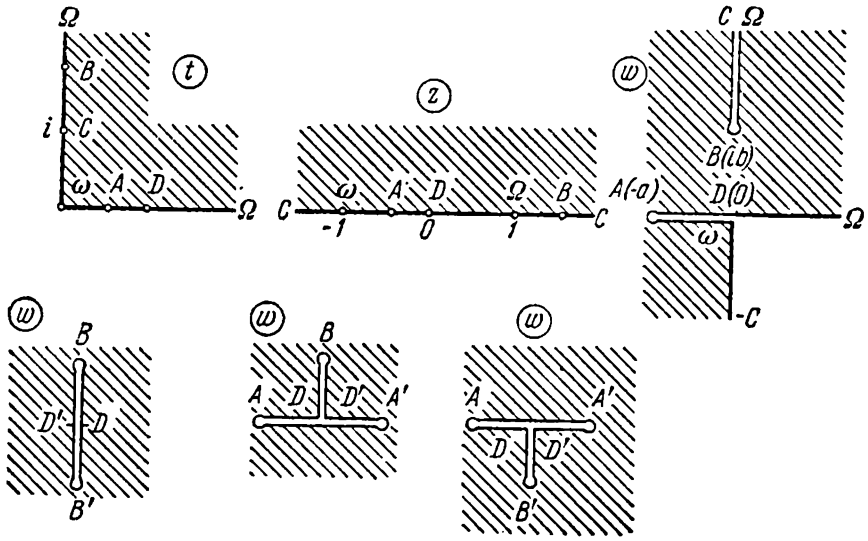

FIG. 94

1254. (a) n odd. F_z has n-sheets and has n algebraic branch points of the $(n-1)$th order above the points $z = \omega^k$ ($\omega = e^{\pi i/n}$, $k = 1, 2, \ldots, n$), F_w has $2n$ sheets and n^2 algebraic branch points of the first order, n over each of the points $w = \eta^\nu \sqrt[n]{(1/4)}$ ($\eta = e^{2\pi i/n}$, $\nu = 0, 1, 2, \ldots, n-1$), corresponding to the points of F_z above $z = \eta^\nu$, and n algebraic branch points of the first order above $w = \infty$, corresponding to the algebraic branch points of F_z above the $z = w^k$. The mapping is shown in Fig. 95 (for the investigation use is made

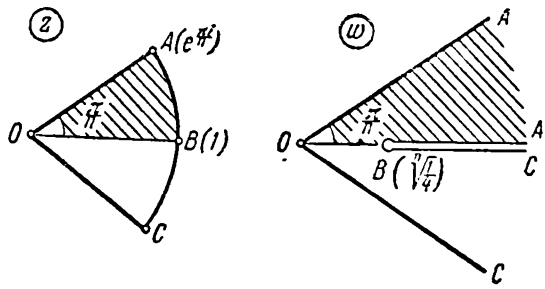

FIG. 95

of the substitutions $\zeta = z^n$, $W = w^n$). The function $w(z)$ maps the circle $|z| < 1$ onto the w-plane with n radial cuts from the points $\eta^\nu \sqrt[n]{(1/4)}$; this is a well-known extremal function from the theory of single-valued conformal mappings†.

† See G. M. GOLUZIN, *The geometrical theory of functions of a complex variable*, (*Geometricheskaya teoriya funktsii kompleksnogo peremennogo*), page 184, Gostekhizdat. 1952.

(b) n even. If $n = 2m$, then $w(z)$ factorises into two functions: $w_{1,2}^m = \pm \frac{z^m}{1+z^{2m}}$. In order to construct F_z take n "half-sheets" $|z| < 1$ and n "half-sheets" $|z| > 1$. Let us denote them by H_k^0 and H_k^+ respectively, and the boundary arcs defined by the points ω^k, by γ_k. Let us attach to H_k^+ along γ_k the half-sheet H_1^0, along γ_{k+1} the half-sheet H_2^0, along γ_{k+2} the half-sheet H_3^0 and so on cyclically. This also determines the order of attachment F_w of the $2n$ sheets representing the w-plane with n radial sections indicated above.

1255. $w = z^{2/3}$, $z = 0$ and $z = \infty$ are algebraic branch points of the second order.

1256. $w = z^{n_1/m_1}$ $\left(\frac{n_1}{m_1}\right.$ is an irreducible fraction, equal to $\left.\frac{n}{m}\right)$, $z = 0$ and $z = \infty$ are algebraic branch points of the (m_1-1)th order.

1257. $z = 0$ and $z = \infty$ are algebraic branch points of the first order; $z = 1$ is an essential singularity of one of the two branches of the function.

1258. $w = 1 - \frac{z}{3!} + \frac{z^2}{5!} - \ldots$ is an integral function; $z = \infty$ is an essential singularity.

1259. $z = 0$ and $z = \infty$ are algebraic branch points of the first order; $z = 1$ is an essential singularity of one of the two branches of the function, the limit of its simple poles at the points $z_k = \left(\frac{1-a_k}{1+a_k}\right)^2$, $a_k = \frac{\pi}{2} + \pi k$ $(k = 0, \pm 1, \ldots)$. By problem 638, the domain of indeterminateness at the point $z = 1$ is identical with the whole plane.

1260. If $n = 0$, then $z = 0$ and $z = \infty$ are removable singularities and $w \equiv 1$; if $n < 0$, then $z = 0$ and $z = \infty$ are logarithmic branch points where $\lim\limits_{z\to 0} w = \lim\limits_{z\to\infty} w = 1$ and $z = 1$ is an essential singularity for one of the branches of the function; if $n = 1$, then $w = z$; if $n > 1$, then $z = 0$ and $z = \infty$ are logarithmic branch points, the domain of indeterminateness of $w(z)$ at these points coinciding with the whole extended w-plane.

1261 and **1262.** $z = 0$ and $z = \infty$ are logarithmic branch points with the domain of indeterminateness for $w(z)$ coinciding with the extended w-plane.

1263. $z = 1$ and $z = \infty$ are logarithmic branch points where $\lim\limits_{z\to 1} w = \lim\limits_{z\to\infty} w = \infty$ and $z = 0$ is a simple pole for all the branches of $w(z)$ exeept one.

1264. $z = 0$ and $z = \infty$ are logarithmic branch points and $\lim\limits_{z\to 0} w = \lim\limits_{z\to\infty} w = \infty$.

1265. The branch points are the same as for $\sin^{-1} z$ (that is, an infinite set of algebraic branch points of the first order above $z \neq \pm 1$ and two logarithmic branch points above $z = \infty$, $\lim\limits_{z\to\infty} w = 0$; $z = 0$ is a pole of the first order for all the branches of the function except one).

1266. $z = \pm i$ are logarithmic branch points, $\lim\limits_{z\to\pm i} w = \infty$; $z = 0$ is a pole of the second order for all the branches of the function.

1267. The surfaces $w(z)$ and $z(w)$ are the same as for the logarithmic function (logarithmic branch points above the points 0 and ∞). The mapping is easily obtained by means of the parametric representation $z = e^{\zeta}$, $w = e^{a\zeta}$.

1268. The Riemann surface for $w(z)$ has an infinite number of sheets with one logarithmic branch point above each of the points $z = a$, $z = b$ and two logarithmic branch points above $z = \infty$. The surface is obtained by joining together an infinite number of sheets of the z-plane with cuts going from the points $z = a$, $z = b$ to ∞. These sheets correspond to the single-valued branches of the function $w + 2\pi in$ $(n = 0, \pm 1, \pm 2, \ldots)$. On going round the points $z = a$ and $z = b$ these branches pass successively into one another and this determines the character of the attachment of the sheets.

1269. The Riemann surface for $w(z)$ has an infinite number of sheets with one logarithmic branch point above each of the points $z = a$, $z = b$, $z = c$ and three logarithmic branch points above $z = \infty$. The construction is the same as the preceding.

1270. The Riemann surface for $w(z)$ has an infinite number of sheets with one logarithmic branch point above each of the points $z = k\pi$ $(k = 0, \pm 1, \pm 2, \ldots)$. As sheets, z-planes with cuts going from the points $z = k\pi$ to ∞ (for example, along vertical straight lines) can be taken. Two sheets are joined together simultaneously along all the cuts, on one side, as in the construction of the surface of the logarithmic function. At ∞ there is a transcendental singularity, the limit point of the logarithmic branch points.

1271. (1) In every connected part of the Riemann surface of the function $\zeta = \phi(z)$ above the z-plane to which corresponds a connected part of the Riemann surface of the inverse function $z = \phi^{-1}(\zeta)$ above G_ζ, $w(z)$ represents a single analytic function; (2) In every connected part of the Riemann surface of the function $\zeta = \phi(z)$, located above G_z (this is the domain G_ζ, transferred to the z-plane), $w(z)$ represents a single analytic function; (3) The same as in part (2). The particular case of $w(z)$ indicated in the conditions of the problem always represents a single analytic function.

1272. $w(z)$ consists of the two analytic functions $\pm z$.

1273. $w(z)$ consists of the two analytic functions $\pm z^{2/3}$.

1274. $w(z)$ consists of the p analytic functions $\omega^k z^{m_1/n_1}$.

$$\left(\omega = e^{2\pi i/p},\quad p = H.C.F \text{ of } (m,n);\ k = 0, 1, \ldots, p-1;\ m_1 = \frac{m}{p},\ n_1 = \frac{n}{p}.\right)$$

1275. $w(z)$ consists of the n integral functions $\omega^k e^{z/n}$ $(\omega = e^{\frac{2\pi i}{n}}$, $k = 0, 1, \ldots, n-1)$.

1276. $w(z)$ is a single n-valued function with algebraic branch points of the $(n-1)$th order above $z = k\pi$ $(k = 0, \pm 1, \ldots)$. At ∞ it has a non-isolated singularity, the limit of the algebraic branch points.

1277. $w = n \operatorname{Log} z + 2\pi ik$ $(k = 0, 1, \ldots, n-1)$ is n distinct analytic functions.

1278. $w(z)$ consists of the functions $z + 2\pi ik$ $(k = 0, \pm 1, \ldots)$.

1279. A single infinitely many valued function with logarithmic branch points above $z = 0, \pm 1, \infty$.

1280. $w(z)$ is a single infinitely many valued function with one logarithmic branch point above each of $z_k = 2\pi ik$ $(k = 0, \pm 1, \ldots)$. At ∞ it has a non-isolated singularity, the limit of the logarithmic branch points. The Riemann surface of the function $w(z)$ is simply connected and is obtained by joining together an infinite number of sheets of the w-plane with cuts (no two having

points in common) going from the z_k to ∞ (two sheets are joined together simultaneously along all the cuts, but only along a definite side of them).

1281. The Riemann surface is the same as in problem **1280**, only with logarithmic branch points above $z = \pi k$ ($k = 0, \pm 1, \ldots$).

1282. The Riemann surface is the same as in problem **1280**, only with logarithmic branch points above $z = \dfrac{\pi k}{2}$ ($k = 0, \pm 1, \ldots$).

1283. $w(z)$ consists of the functions $\pm z + 2k\pi$ ($k = 0, \pm 1, \ldots$).

1284. $w(z)$ consists of the functions $z + k\pi$ ($k = 0, \pm 1, \ldots$).

1285. (1) Let $r_1 = \dfrac{m_1}{n_1}$, $r_2 = \dfrac{m_2}{n_2}$, $r = r_1 r_2 = \dfrac{m}{n}$ be irreducible fractions and $p = (m_1, n_2)$ (the greatest common divisor of m_1 and n_2), $q = (m_2, n_1)$. Then $(z^{r_1})^{r_2}$ consists of p distinct n-valued analytic functions, equal to $\sqrt[p]{1}.z^r = \sqrt[p]{1}.(\sqrt[n]{z})^m = \sqrt[p]{1}.\sqrt[n]{(z^m)}$, and $(z^{r_2})^{r_1}$ consists of the q functions $\sqrt[q]{1}.z^r$. One of them, namely z^r, always belongs to both cases. In particular, $(z^{2/3})^{3/2} = \pm z$, $(z^{3/2})^{2/3} = \sqrt[3]{1}.z$.

(2) Let $r_1 = m_1/n_1$; $r_2 = m_2/n_2$, $r = r_1 + r_2 = m/n$ be irreducible fractions and $p = (n_1, n_2)$. Then $z^{r_1} z^{r_2}$ consists of p distinct n-valued analytic functions equal to $\sqrt[p]{1}.z^r$.

(3) Using the preceding notation let $N = \dfrac{n_1 n_2}{p}$, $n_1' = \dfrac{n_1}{p}$ and $n_2' = \dfrac{n_2}{p}$. Then $w(z)$ consists of p distinct analytic N-valued functions with the parametric representation: $z = t^N$, $w = \sqrt[p]{1}\,(t^{m_1 n_2'} + t^{m_2 n_1'})$. The Riemann surfaces of all these functions above the z-plane are the same as that of the function $\sqrt[N]{z}$.

1286. Let $N =$ L.C.M. of m and n, $p = (m,n)$, $q = \left(\dfrac{m+n}{p}, \dfrac{mn}{p}\right)$ and $\dfrac{1}{m} + \dfrac{1}{n} = \dfrac{\mu}{\nu}$. Then $w(z)$ is a single N-valued analytic function having N/n algebraic branch points of the $(n-1)$th order above $z = 0$, N/m algebraic branch points of the $(m-1)$th order above $z = 1$ and N/q algebraic branch points of the $(\nu-1)$th order above $z = \infty$.

1287. One nm-valued analytic function having one algebraic branch point of the $(n-1)$th order above $z = 1$, n algebraic branch points of the $(m-1)$th order above $z = 0$ and one algebraic branch point of the $(nm-1)$th order above $z = \infty$.

1288. Two distinct four-valued functions differing in sign with the same Riemann surface as the function $\sqrt[4]{z}$. These functions have each one branch for which the point $z = 1$ is a pole of the first order.

1289. An infinitely many valued function with one algebraic branch point of the $(n-1)$th order above $z = 1$ and n logarithmic branch points above each of the points $z = 0$, $z = \infty$. In order to construct the surface it is necessary to join n surfaces for Log z with the cut $[1, \infty)$ on one of the sheets of each. The curves $\varrho^n \sin n\theta = k\pi$ ($k = 0, \pm 1, \pm 2, \ldots$) divide the w-plane into domains corresponding to the half-planes $y \gtrless 0$ (Fig. 96, for $n = 2$, $\zeta = w^2$ is an auxiliary plane).

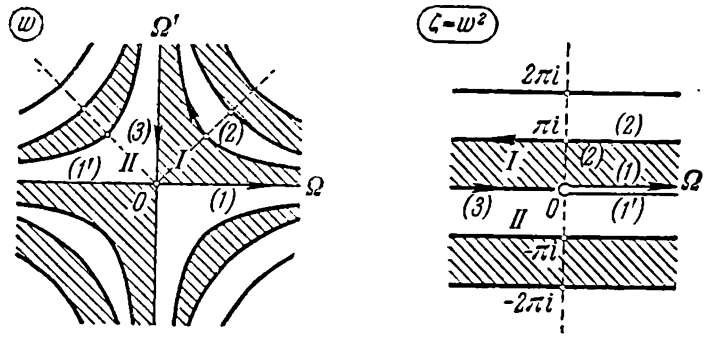

FIG. 96

1290. An infinitely many valued function with one logarithmic branch point above $z = 1$ and an infinite number of only logarithmic branch points above $z = 0$ and $z = \infty$. The Riemann surface is obtained by joining an infinite number of surfaces for Log z with the cut $[1,\infty)$ on one of the sheets. Above $z = 0$, $z = \infty$ the surface has only logarithmic branch points though they are infinitely many, and above $z = 1$ ordinary points and one logarithmic branch point. The curves $e^u \sin v = (2k+1)\pi$ and $v = 2k\pi$ ($k = 0, \pm 1, \pm 2, \ldots$) divide the w-plane into domains, each corresponding to the z-plane with the cut $-\infty < x \leqslant 0$, $y = 0$ and the complementary cut $1 \leqslant x < \infty$, $y = 0$ for the domains whose boundary contains one of the straight lines $v = 2k\pi$ (Fig. 97; $\zeta = e^w$ is the auxiliary plane).

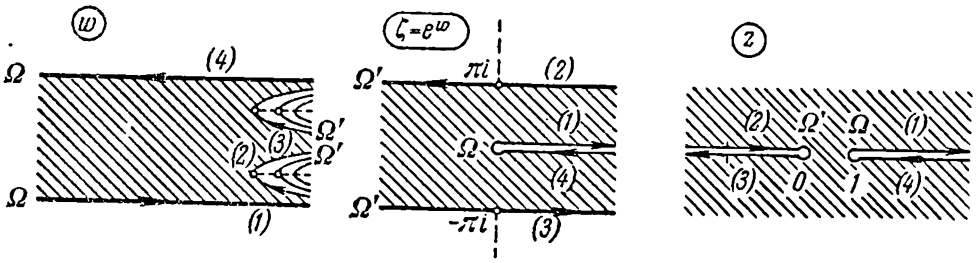

FIG. 97

1291. An infinitely many valued function with the same Riemann surface as Log Log z.

1292. The Riemann surface has infinitely many sheets and has an algebraic branch point of the $(n-1)$th order above $z = 0$, only algebraic branch points of the first order above $z = \pm 1$ and $2n$ logarithmic branch points above $z = \infty$. In order to construct the surface it is necessary to join n surfaces for $\sin^{-1} z$ with the cut $[0,1]$ on one of the sheets of each of them. The curves $n\theta = k\pi/n$ and $\varrho^n \cos n\theta = k\pi$ ($k = 0, \pm 1, \ldots$) divide the w-plane into domains corresponding to the half-planes $y \gtrless 0$ (Fig. 98, where $n = 2$, $\zeta = w^2$ is the auxiliary plane).

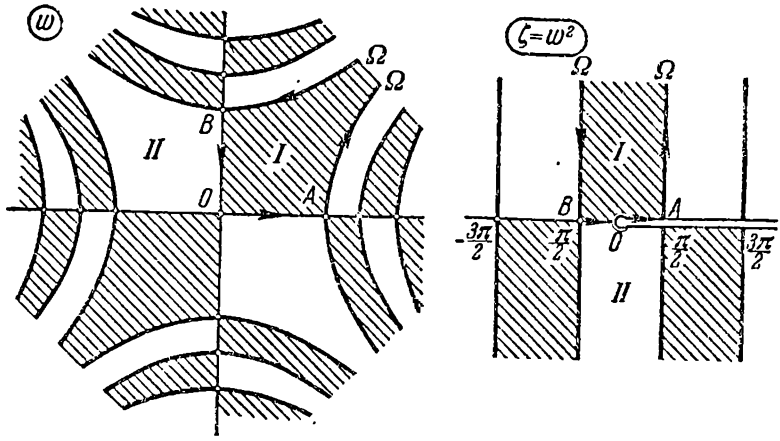

FIG. 98

1293. The Riemann surface is obtained by joining an infinite number of the surfaces for Log z with the cuts $[0, 1/e],[e, \infty)$ on one of the sheets (the surfaces for Log z are joined together in pairs, first along one and then along the other of these cuts). The surface has above $z = 0$, $z = \infty$ an infinite set of logarithmic branch points and above $z = 1/e$, $z = e$ ordinary points and infinitely many algebraic branch points of the first order. The curves Im sin $w = \cos u \sinh v = (2k+1)\pi$ and $u = \pi/2+k\pi$ $(k = 0, \pm 1, \pm 2, \ldots)$ divide the w-plane into domains each corresponding to the z-plane with the cut $-\infty < x < 0$, $y = 0$ and with the two complementary cuts: $0 \leqslant x \leqslant \frac{1}{e}$ $y = 0$ and $e \leqslant x < \infty$, $y = 0$ for the domains whose boundary contains one of the straight lines $u = \frac{\pi}{2} + k\pi$ (Fig. 99; $\zeta = \sin w$ is the auxiliary plane).

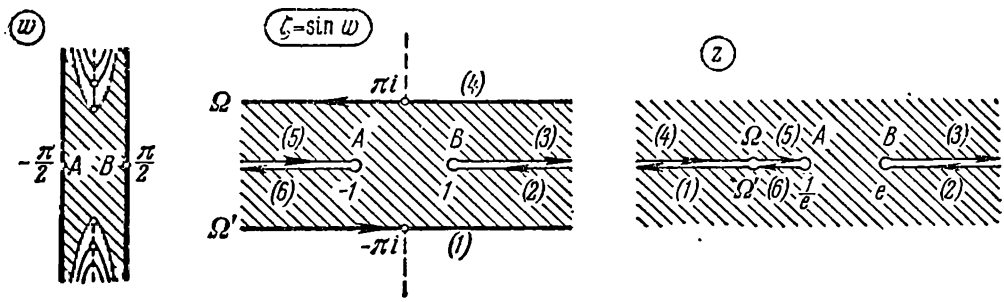

FIG. 99

1294. An infinitely many valued function. Its Riemann surface is obtained by joining together infinitely many specimens of the Riemann surface of the function $\sqrt[n]{(z)}-1$, provided with a cut along the ray $1 \leqslant x < \infty$, $y = 0$

on one of the sheets. The attachment proceeds as in the construction of the Riemann surface of the logarithmic function.

1295. An infinitely many valued function with one only algebraic branch point of the first order above $z = 0$ and $z = \infty$ and two logarithmic branch points above $z = 1$.

1296. An infinitely many valued function with two logarithmic branch points above $z = -1$ and an infinite number of only algebraic branch points of the first order above $z = 0$ and $z = \infty$.

1297. If $\alpha = m/n$, then $w = \frac{m}{n} \operatorname{Log} z + \frac{2\pi i}{n} k$ $(k = 0, 1, \ldots, m-1)$ are m distinct analytic functions; if α is irrational then $w = \alpha \operatorname{Log} z + 2\pi i k$ $(k = 0, \pm 1, \ldots)$ are infinitely many distinct functions.

1298. Two distinct infinitely many-valued functions with the same Riemann surface as Log z.

1299. Two distinct analytic functions equal respectively to 2 Log z and 2 Log $(-z)$.

1300. $w = \frac{\pi}{2} + 2\pi k$ and $w = 2 \cos^{-1} z - \frac{\pi}{2} + 2\pi k$ $(k = 0, \pm 1, \ldots)$.

1301. $w = \frac{\pi}{2} + \pi k$ $(k = 0, \pm 1, \ldots)$.

1302. $w = i\,(\pi/2 + \pi k)$ $(k = 0, \pm 1, \ldots)$.

1303. w is an infinitely many valued function with the same Riemann surface as Log Log z (see, problem **1290**). If $\zeta = \varrho e^{i\theta} = \operatorname{Log} z = \log r + i\phi$ $(\phi = \operatorname{Arg} z)$, then $w = e^{i \operatorname{Log} \zeta} = e^{i(\log \varrho + i\theta + 2\pi i k)}$. For $f_0(z) = e^{-\theta + i \log \varrho}$ $(0 \leqslant \theta \leqslant 2\pi)$ given in the problem the sets of limiting values represent respectively: (1) and (2) the circle $|w| = e^{-\pi}$;

(3) the circles $|w| = e^{-\pi/2}$ as $\phi \to +\infty$ and $|w| = e^{-3\pi/2}$ as $\phi \to -\infty$;

(4) the ring $e^{-3\pi/2} \leqslant |w| \leqslant e^{-\pi/2}$.

For other groups of branches the factor $e^{-2\pi k}$ $(k = \pm 1, \pm 2, \ldots)$ is obtained.

1304. At all the points $w(z)$ represents a single analytic function if $|a| < 1$; if $|a| \geqslant 1$, then in parts (1) and (3) the function separates into n, and in parts (2) and (4) into an infinite number of distinct analytic functions.

1305. If $|a| < 1$, then in both cases $w(z)$ represents a single analytic function; if $|a| \geqslant 1$, then in part (1) there will be n and in part (2) an infinite number of distinct analytic functions.

1306. (1) Let $z = re^{i\varrho}$ and $\zeta = \varrho e^{i\theta} = \operatorname{Log} z$. Then $w(z)$ consists of many-valued analytic functions, equal respectively to $\chi(z)\, e^{i \log \varrho - \theta + 2\pi k}$ $(|z| < 1$; $\varrho = \sqrt{(\log^2 r + \phi^2)}$, $\theta = \arg \log r + i\phi)$, where $-\infty < r < \infty$ and $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$; $(k = 0, \pm 1, \ldots)$. They each have one logarithmic branch point above $z = 0$, in the neighbourhood of which the set of indeterminateness is a ring (in particular, for $k = 0$ the ring is $\chi(0) e^{-3\pi/2} \leqslant |w| \leqslant \chi(0)\, e^{-\pi/2}$.

(2) Let $z - 1 = re^{i\phi}$ and $\zeta = \varrho e^{i\theta} = \operatorname{Log}(z-1)$. Then $w(z)$ consists of single-valued analytic functions equal respectively to $\chi(z) e^{i \log \varrho - \theta + 2\pi k}$ $(|z| < 1$; $\varrho = \sqrt{(\log^2 r + (\phi + 2\pi n)^2)}$, $\theta = \arg\, [\log r + i(\phi + 2\pi n)]$, where $\frac{\pi}{2} < \phi < \frac{3\pi}{2}$, $0 < \theta < 2\pi$; $k, n = 0, \pm 1, \ldots)$. For different k, n these are different analytic functions. If $\chi(1) = \lim_{z \to 1} \chi(z)$ exists, then the set of

indeterminateness as $z \to 1$ is a circle (in particular, for $k = n = 0$ it is the circle $|w| = \chi(1)\, e^{-\pi}$).

1307. (1) If $f(z)$ has no zeros of odd order, then $\sqrt{(f(z))}$ factorises into two integral functions. If $a_1, a_2, \ldots$ are the zeros of odd order of $f(z)$, then the Riemann surface for $\sqrt{(f(z))}$ has two sheets with algebraic branch points above $a_1, a_2, \ldots$ and above ∞, if $f(z)$ has a pole of odd order there. If $f(z)$ is a transcendental function, then above $z = \infty$ there are two essential singularities of single-valued character, if $f(z)$ has an even number of zeros of odd order, and one of the essential singularities is of two-valued character if the number of such zeros is odd or infinitely great.

(2) If $f(z)$ has the zeros $a_1, a_2, \ldots$ then the Riemann surface for Log $f(z)$ has one logarithmic branch point above each of $a_1, a_2, \ldots$ and no other points. If $f(z)$ has no zeros, then Log $f(z)$ factorises into an infinite number of integral functions differing from one another by terms of the form $2\pi ki$ $(k = 0, \pm 1, \pm 2, \ldots)$.

(3) If $f(z)$ has zeros then the Riemann surface for $[f(z)]^\alpha$ is the same as that for Log $f(z)$ (see part (2)). If $f(z)$ does not have zeros, then $[f(z)]^\alpha$ factorises into an infinite number of integral functions differing from one another by factors of the form $e^{2\pi k\alpha i}$ $(k = 0, \pm 1, \pm 2, \ldots)$.

1308. The two-sheeted circle $|z| < 1$ with algebraic branch points at the zeros of the function $\sum_{n=1}^{\infty} z^{n!}$, in particular at the point $z = 0$.

1309. (1) The two-sheeted circle $|z| < 1$ with a single branch point at $z = 0$ (part of the Riemann surface of the function $\sqrt[1]{z}$ situated above the circle $|z| < 1$);

(2) The part of the Riemann surface of the function Log z situated above the circle $|z| < 1$;

(3) The part of the Riemann surface of the function Log z situated above the ring $\frac{1}{2} < |z| < 2$.

CHAPTER X

1324. (1) At the vertex A_2, $\alpha = 0$; (2) At the vertices A_1 and A_2, $\alpha_1 = \alpha_2 = 0$; (3) At the vertex A_3, $\alpha_3 = 0$; (4) At the vertices A_2 and A_4, $\alpha_2 = \alpha_4 = -1$; (5) At the vertex A_2, $\alpha_2 = -2$, at the vertex A_4, $\alpha_4 = 0$; (6) At the vertices A_2, A_3 and A_4, $\alpha_2 = \alpha_3 = \alpha_4 = 0$; (7) At the vertex A_2, $\alpha_2 = -2$, at the vertices A_4 and A_6, $\alpha_4 = \alpha_6 = 0$; (8) At the vertex A_2, $\alpha_2 = -2$, at the vertex A_4, $\alpha_4 = \alpha - 2$.

1326. It is necessary and sufficient that $\alpha_k = \dfrac{1}{n_k}$ (n_k is a natural number or infinity) and $\sum\limits_{k=1}^{n}\left(1-\dfrac{1}{n_k}\right) = 2$, which is possible only for $n = 4$ with $n_1 = n_2 = n_3 = n_4 = 2$ (that is for a rectangle) and for $n = 3$;

n_1	n_2	n_3	P
1	∞	∞	a strip
2	2	∞	a half-strip
2	3	6	a right-angled triangle
2	4	4	an isosceles right-angled triangle
3	3	3	an equilateral triangle

1327. (1) $w = \dfrac{h}{\pi}\log z + a$, $z = e^{\pi(w-a)/h}$ (a is a real parameter), $z(w)$ is a periodic function with period $\omega = 2\pi i$; the group G is generated by the transformation $T(w) = w + \omega$; its fundamental domain B consists of a doubled strip and one of its boundary sidos;

(2) $w = h/\pi \log[(1+z)/(1-z)] + a$, $z = \tanh[\pi(w-a)/2h]$ (a is a real parameter); $z(w)$ is a periodic function with period $\omega = 2hi$; the group G and its fundamental domain B are the same as in part 1.

1328. $w = \sin^{-1}z$, $z = \sin w$; $z(w)$ is a periodic function with period $\omega = 2\pi$; the group G is generated by the transformations $T(w) = w + \omega$, $S(w) = -w$; its fundamental domain B consists of the strip $0 < u < \pi$ and the boundary half-lines $u = 0$, $u = \pi$, $v \leqslant 0$.

1329†. (1) $w = C\int\limits_0^z z^{-5/6}(1-z)^{-1/2}dz$, where $C = \omega/[B(1/6,\ 1/2)]$ $= (B(p,q) = \int\limits_0^1 x^{p-1}(1-x)^{q-1}dx$ is Euler's integral of the first kind); $z(w)$ is a doubly periodic function with periods 2ω and $2\omega e^{2\pi i/6}$; the group G is generated by the transformations: $T(w) = w + 2\omega$, $S(w) = we^{2\pi i/6}$;

† Diagrams of the fundamental domains are given in Fig. 62, page 292.

its fundamental domain B consists of the doubled triangle and two different boundary sides;

(2) $w = C\int_0^z z^{-\frac{1}{2}}(1-z)^{-\frac{1}{2}}\,dz$, where $C = \omega/[B(1/4, 1/2)]$; $z(w)$ is a doubly periodic function with periods 2ω and $2\omega i$; the group G is generated by the transformations: $T(w) = w+2\omega$, $S(w) = iw$; its fundamental domain B consists of a square with side ω and two boundary sides of one of the triangles forming the square;

(3) $w = C\int_0^z z^{-\frac{2}{3}}(1-z)^{-\frac{2}{3}}dz$ where $C = \omega/[B(1/3, 1/3)]$; $z(w)$ is a doubly periodic function with periods $2hi$ and $2he^{\pi i/6}$, where $h = \omega\sqrt{3}/2$; the group G is generated by the transformations: $T(w) = w+2hi$, $S(w) = we^{2\pi i/3}$; its fundamental domain B consists of the doubled triangle and two different boundary sides.

1330. (1) The "triangle" with two vertices at the points $w = 0$, $w = d = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta)$ and angles $\pi\alpha$, $\pi\beta$ at these vertices. If $\alpha+\beta < 1$ then the three vertices are finite; if $\alpha+\beta \geqslant 1$, the three vertices lie at infinity; if $\alpha+\beta = 1$, then $d = \pi/\sin\alpha\pi$, and the "triangle" has the shape of a half-strip, oblique, if $\alpha \neq \beta$; in the case $\alpha+\beta = 2$ the sides of the "triangle" which issue from the vertices of the base are parallel, directed in opposite directions and $d = \dfrac{\pi(\alpha-1)}{\sin\pi(\alpha-1)}$; if $\alpha = \beta = \dfrac{3}{2}$, then the "triangle" constitutes the exterior of a straight half-strip (Fig. 100).

(2) A "triangle" with one finite vertex at the point $w = 0$, with the angle $\pi\alpha$ and with two vertices at ∞. Two sides of the "triangle" form rays, issuing from the origin, the third side is a straight line standing at a distance $h = \dfrac{\sin\pi\beta}{\beta}\,\dfrac{\Gamma(\alpha)\Gamma(\beta+1)}{\Gamma(\alpha+\beta)}$ from the origin. In the case $\alpha = 1$ a strip of width π is obtained; in the case $\alpha+\beta = 1$ two sides are parallel and $h = \pi$; in the case $\alpha = 2$ a half-plane is obtained with a cut along the real positive semi-axis and $h = \dfrac{\sin\pi\beta}{\beta(\beta+1)}$, in particular, $h = 4$, if $\beta = -\dfrac{1}{2}$, and $h = \pi$, if $\beta = -1$ (see Fig. 100).

1331. (1) See Fig. 101, (1); (2) see Fig. 101, (2).

1332. (1) $w = \dfrac{2}{\pi}[\sin^{-1}\sqrt{z} - (1-2z)\sqrt{(z-z^2)}]$;

(2) $w = \dfrac{2}{\pi}[\sin^{-1}\sqrt{z} - \sqrt{(z-z^2)}]$;

(3) $w = \dfrac{h}{\pi}\left[\log\dfrac{1+\sqrt{z}}{1-\sqrt{z}} - 2\sqrt{z}\right] = \dfrac{2h}{\pi}(\tanh^{-1}\sqrt{z} - \sqrt{z})$;

(4) $w = \dfrac{2h}{\pi}[\tan^{-1}\sqrt[4]{z} + \tanh^{-1}\sqrt[4]{z} - 2\sqrt[4]{z}]$;

(5) $w = ia\left(-\sqrt{z}\cdot\dfrac{z-3}{2} - 1\right)$.

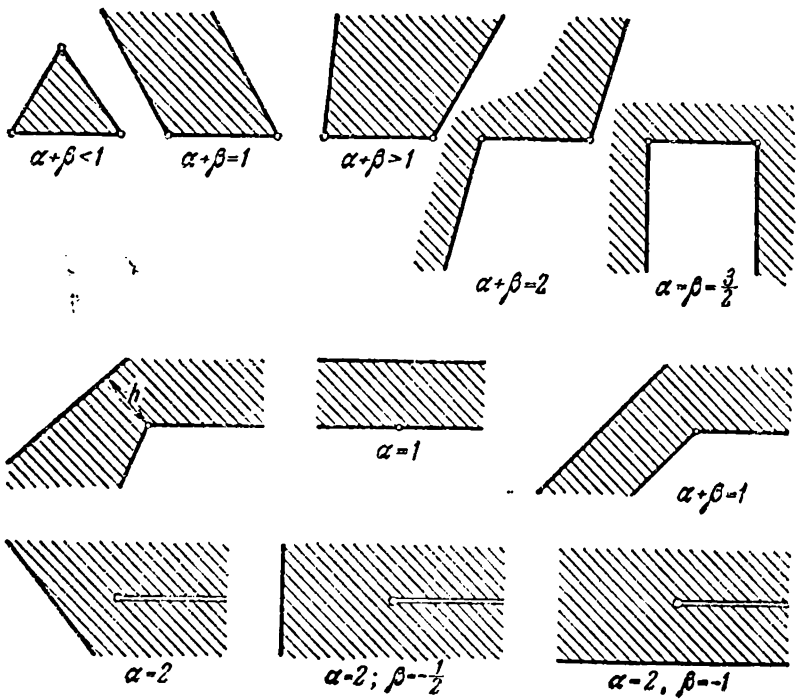

Fig. 100

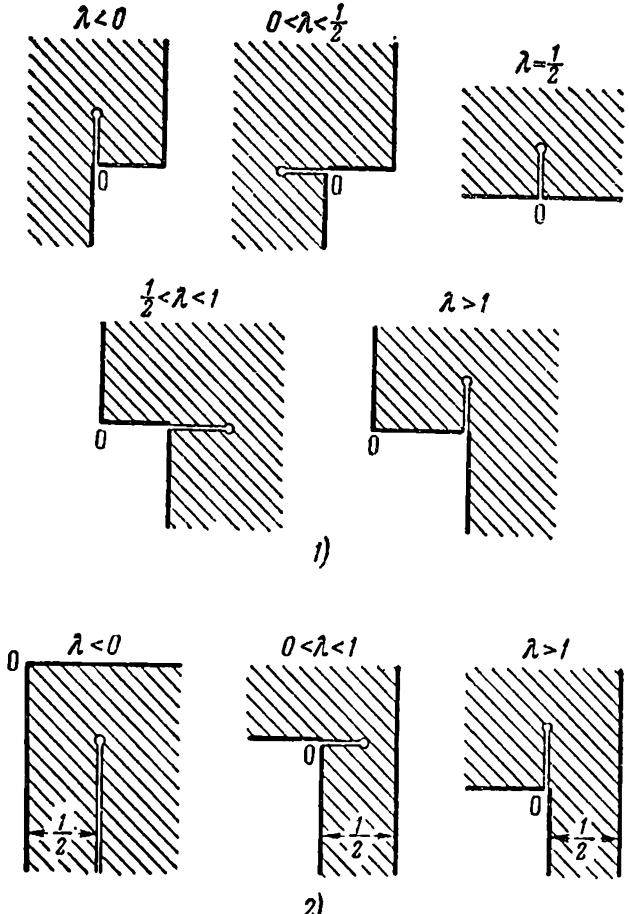

Fig. 101

1333. (1) $w = -\frac{a}{\pi}\int_1^z \frac{dz}{z^{1-\theta}(z-1)^{\theta}}$. If $\theta = \frac{h}{q}$, then $w = \frac{a}{\pi}\sum_{\nu=0}^{q-1}\frac{1}{t_\nu^p} \times$

$\times \log\left(1-\frac{t}{t_\nu}\right)$, where $t = \left(\frac{z-1}{z}\right)^{\frac{1}{q}}$ and $t_\nu = e^{\frac{2\nu\pi i}{q}}$ $(\nu = 0, 1, \ldots, q-1)$.

(2) $w = \frac{a}{\pi\theta}\int_1^z \left(\frac{z-1}{z}\right)^{\theta} dz$. If $\theta = p/q$, then $w = \frac{a}{\pi}\left[-\frac{t^p}{\theta(t^q-1)}\right.$

$\left. + \sum_{\nu=0}^{q-1} t_\nu^p \log\left(1-\frac{t}{t_\nu}\right)\right]$, where t and t_ν have the same values as in part (1).

1334. $w = \frac{a}{\zeta}$, where $\zeta = \frac{1}{\pi}\left(\frac{2t}{1-t^2} + \log\frac{1-t}{1+t}\right)$, $t = \sqrt{\left(\frac{z-1}{z}\right)}$.

1335. $w = \frac{2h}{\pi}\left(\sqrt{(z^2-1)} + \sin^{-1}\frac{1}{z}\right)$.

1336. $w = \frac{2d}{B\left(\frac{\alpha}{2}, 1-\alpha\right)}\int_0^z t^{\alpha-1}(1-t^2)^{-\alpha}\,dt$.

1337. (1) $w = \frac{2}{\pi}\left\{H\tan^{-1}\sqrt{\left(\frac{z}{a-z}\right)} + h\tanh^{-1}\frac{h}{H}\sqrt{\left(\frac{z}{a-z}\right)}\right\}$, where

$a = 1+\frac{h^2}{H^2}$;

(2) $w = \frac{2}{\pi}\left\{H\tan^{-1}\frac{z}{\sqrt{(a^2-z^2)}} + h\tanh^{-1}\frac{h}{H}\frac{z}{\sqrt{(a^2-z^2)}}\right\}$, where

$a^2 = 1+\frac{h^2}{H^2}$;

(3) $w = C\int_0^z \frac{\sqrt{t}dt}{(t-1)(t-a)(t+b)}$, where C, a, b are determined from the equations:

$$\frac{C\pi}{(a-1)(a+b)} = h_1,\ \frac{C\pi\sqrt{a}}{(a-1)(a+b)} = h_2,\ \frac{C\pi\sqrt{b}}{(b+1)(a+b)} = h_3.$$

1338. (1) $w = \frac{n}{B\left(\frac{1}{n}, 1-\frac{2}{n}\right)}\int_0^z (1-t^n)^{-\frac{2}{n}}dt$;

(2) $w = C\int_z^1 \frac{(1-t^n)^{\frac{2}{n}}}{t^2}\,dt + 1$, $c_{-1} = C = \frac{2^{-2/n}n\sin\pi/n}{B\left(\frac{1}{n}+\frac{1}{2}, \frac{1}{2}\right)}$.

1339. (1) $w = \dfrac{5.2^{2/5}}{B\left(\frac{1}{10}, \frac{1}{5}\right)} \int\limits_0^z (1-t^5)^{-4/5}(1+t^5)^{2/5}\,dt;$

(2) $w = C\int\limits_z^1 \dfrac{(1-t^5)^{4/5}(1+t^5)^{-2/5}}{t^2}\,dt+1,\ c_{-1} = C = \dfrac{2^{-2/5}\Gamma(7/10)}{\Gamma(9/10)\Gamma(4/5)}.$

1340. Onto a many pointed star with angles $\pi-\dfrac{2\pi}{n}-\lambda\pi$ and $\pi+\lambda\pi$ alternately, with centre at the coordinate origin and one of the vertices of the first type of angle at the point

$$w = \frac{2^{-2/n}}{n}\,\frac{\Gamma\left(\frac{1-\lambda}{2}-\frac{1}{n}\right)\Gamma\left(\frac{1+\lambda}{2}\right)\sin\frac{\pi(1+\lambda)}{2}}{\sin\pi/n}.$$

1341. $w = CC_0\int\limits_i^z \left\{z\prod\limits_{k=1}^{m-1}\left(z^2-\tan^2\frac{\pi k}{n}\right)\right\}^{-2/n} dz$, if $n = 2m$, and

$$w = CC_1\int\limits_i^z \left\{z\prod\limits_{k=1}^{m}\left(z^2-\tan^2\frac{\pi k}{n}\right)\right\}^{-2/n} dz,\ \text{if } n = 2m+1, \text{ where}$$

$$C = \frac{n}{B\left(\frac{1}{n}, 1-\frac{2}{n}\right)},\ C_0 = -2i(2n)^{-2/n}\,e^{-i\pi/n} \text{ and } C_1 = -2i.2^{-2/n}.$$

1342. The parameters are determined by means of equation (3) for the b_k, from the equalities $|f(b_k)| = l_k$ $(k = 1, 2, \ldots, n)$ and the direction of one of the sides of the star. One value of α_k is chosen arbitrarily.

1343. $w = \dfrac{C(z-1)^{2\alpha}(z+1)^{2\alpha-2}}{z},\quad C = \dfrac{h}{4}\alpha^{-2\alpha}(1-\alpha)^{2\alpha-2}.$

1345. The parameters are determined from the values of $|f(b_i)|$, $|f(\alpha_s)|$, known from P and one of the directions of the sides of P being given. Three of the parameters (a_k, b_l, c_j, d_s) are chosen arbitrarily. If one of the parameters α_k or c_j is equal to infinity, then (4) and (5) apply if the factor and term with this parameter are neglected. If one of the parameters b_i or d_s is equal to ∞ then (4) and (5) remain in force without alteration.

1347. $w = C(z+1)^{\alpha_1}(z-1)^{\alpha_2},\ C = h(b+1)^{-\alpha_1}(1-b)^{-\alpha_2},\ b = \dfrac{\alpha_1-\alpha_2}{\alpha_1+\alpha_2}.$

1348. $w = C\sqrt{\left(\dfrac{z^2-a^2}{1-a^2z^2}\right)},\ C = \sqrt{(hH)},\ a = \sqrt{(h/H)}.$

1349. $w = Cz^{1-2\alpha}(z^2-1)^{\alpha},\ C = hb^{2\alpha-1}(1-b^2)^{-\alpha},\ b = \sqrt{(1-2\alpha)}.$

1350. $w = \sqrt{(hH)}\left(\dfrac{z+a}{1+az}\right)^{\alpha_1}\left(\dfrac{z-b}{1-bz}\right)^{\alpha_2}$, where a and b are determined from

the system of equations

$$a^{\alpha_1}b^{\alpha_2} = \gamma/(h/H),\ \alpha_1\left(\frac{1}{a}-a\right) = \alpha_2\left(\frac{1}{b}-b\right).$$

1351. (1) $w = [T_n(z)]^{1/n}$, where $T_n(z) = \frac{1}{2}[(z+\sqrt{(z^2-1)^n})+(z-\sqrt{(z^2-1)^n})]$ are Chebyshev polynomials;

(2) $w = \left[\frac{T_n'(z)\sqrt{(z^2-1)}}{n}\right]^{1/n}$.

1352. The parameters are determined from the values of $\operatorname{Re} f(b_i)$ and $\operatorname{Re} f(d_s)$ known from P and the position of one of the sides of P being given. Three of the parameters (a_k, b_i, c_j, d_s) are chosen arbitrarily. If one of the parameters a_k or c_j equals ∞, then (6) and (7) still apply if the corresponding terms are neglected. If one of the parameters b_i or d_s equals ∞, then (6) and (7) apply without alteration (see the answer to problem **1345**).

1353. The parameters are determined from the values of $\operatorname{Re} f(b_i)$ and $\operatorname{Re} f(d_s)$. Three of the parameters a_k, b_i, c_j, d_s, are chosen arbitrarily. In formula (10) two of the parameters a_k, b_i, c_j, d_s are chosen arbitrarily.

1354. The parameters are determined from the values of $\operatorname{Re} f(b_i)$ and the position of one of the sides of P. Two of the parameters are chosen arbitrarily.

1355. The parameters are determined from the values of $\operatorname{Re} f(b_i)$ and $\operatorname{Re} f(d_s)$ and the position of one of the sides of P. Three parameters are chosen arbitrarily. In formula (11) two parameters corresponding to the vertices are chosen arbitrarily.

1356. (1) $w = \frac{h_1}{\pi}\log(z+1)+\frac{h_2}{\pi}\log(z-1)+C$,

$C = -\frac{h_1}{\pi}\log(b+1)-\frac{h_2}{\pi}\log(1-b)$, $b = \frac{h_1-h_2}{h_1+h_2}$;

(2) $w = \frac{h_1}{\pi}\log(z^2-1)+\frac{h^2}{\pi}\log z+C$, $C = -\frac{h_1}{\pi}\log(1-b^2)-\frac{h_2}{\pi}\log b$,

$b = \sqrt{\left(\frac{h^2}{2h_1+h_2}\right)}$;

(3) $w = \frac{h_1}{\pi}\log\frac{z+a_1}{1+a_1z}+\frac{h_2}{\pi}\log\frac{z-a_2}{1-a_2z}$. The parameters a_1, a_2 are determined from the system of equations: $a_1^{h_1}a_2^{h_2} = e^{-\pi}$,

$h_1\left(\frac{1}{a_1}-a_1\right) = h_2\left(\frac{1}{a_2}-a_2\right)$;

(4) $w = f\left[\frac{1}{2}\left(z+\frac{1}{z}\right)\right]$, where $f(z)$ is the mapping of part (3);

(5) $w = \frac{h}{\pi}\log T_n(z)$, where $T_n(z)$ are Chebyshev polynomials (see the answer to problem **1351**);

(6) $n = \frac{h}{\pi}[\log(z-1)+z]$;

(7) $w = \log\dfrac{z+1}{1-z} + Az$, $A = \dfrac{2}{b^2-1}$, where b is determined from the equation $\log\dfrac{b+1}{b-1} + \dfrac{2b}{b^2-1} = d$;

(8) $w = \log(z+1) - Az^2 - z + \text{const}$, where $A = \dfrac{1}{2a}$ and a is determined from the equation $\log a + \dfrac{1}{2}\left(a - \dfrac{1}{a}\right) + d = 0$. In particular, if $d = 0$, then $a = 1$;

(9) $w = h/\pi \log(z-a) + C/(z-a) + Az + \text{const}$, where $A = h/2\pi a$, $C = \dfrac{h}{2\pi a}(1-a^2)$ and a is determined from the equations $\log\dfrac{1-a}{1+a} + \dfrac{2}{a} = \dfrac{\pi d}{h}$. In particular, if $h = 0$ then $a = 0$, $A = C = d/4$.

1357. The upper half-plane $\operatorname{Im} z > 0$ is mapped onto the rectangle with vertices $\pm K$, $\pm K + iK'$, corresponding to the points ± 1, $\pm\dfrac{1}{k}$. In the ϕ-plane the same half-plane corresponds to the vertical half-strip with the base $(-\pi/2, \pi/2)$, the points given above corresponding to the points $\pm\pi/2$, $\pm\pi/2 + i\log\dfrac{1+k'}{k}$. The whole z-plane with the cuts $(-\infty, -1]$, $[1, \infty)$ is mapped onto the rectangle with vertices $K \pm iK'$, $-K \pm iK'$, with the corresponding pairs of points $1/k$ and $-1/k$ at the sides of the cuts. In the ϕ-plane a vertical strip is obtained passing through the points $\pm\pi/2$ (Fig. 102).

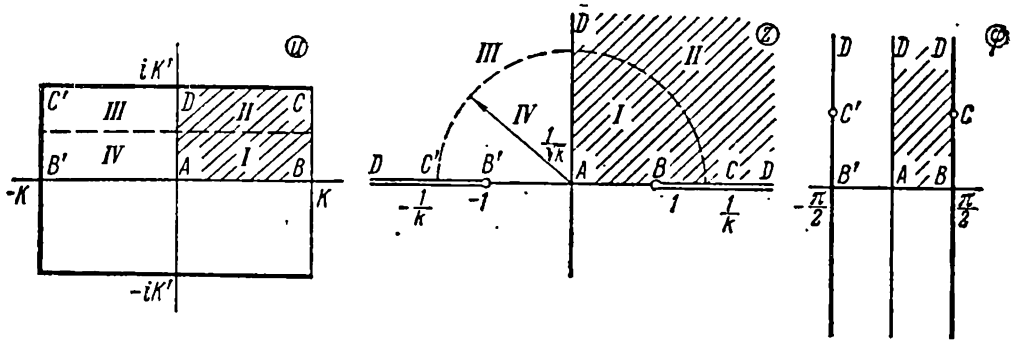

FIG. 102

1358. $w = \lambda\displaystyle\int_0^z \frac{dz}{\sqrt{[(1-z^2)(1-k^2z^2)]}}$, $z = \operatorname{sn}\left(\dfrac{w}{\lambda}, k\right)$, where λ and k are determined from the relations $K'/K = b/a$, $a = \lambda K$.

1359. See Fig. 103. The inverse function is $z = \operatorname{cn}(w, k)$.

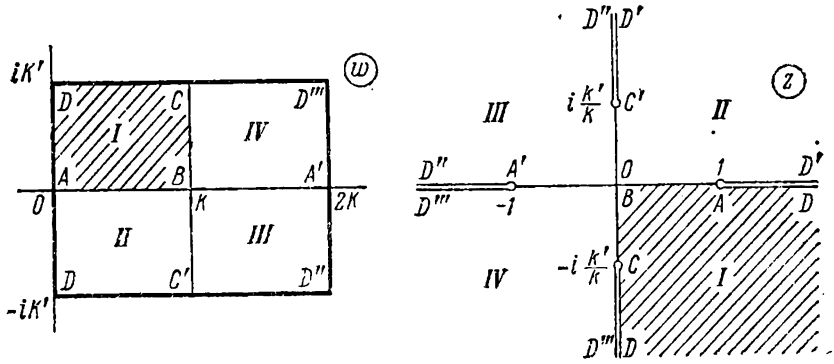

FIG. 103

1360. See Fig. 104. The inverse function is $z = \operatorname{dn}(w, k)$.

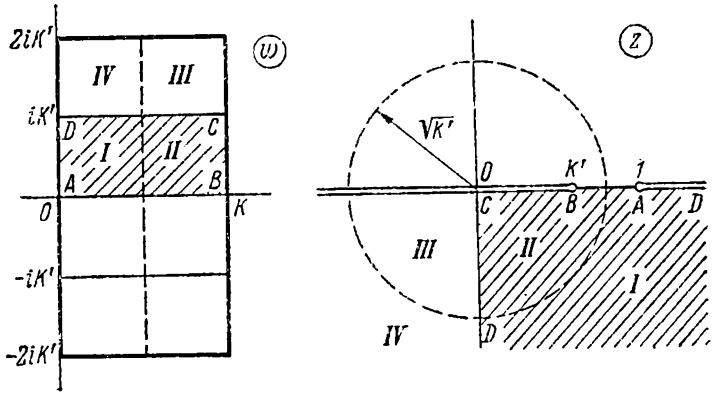

FIG. 104

1362. The mapping of the z_1-plane onto the u_1-plane is shown in Fig. 105 (the w_1-plane on this same diagram refers to the answer to problem **1363**).

1363. The relation between the z_1-plane and the w_1-plane is shown in Fig. 105. The expressions for l and h are indicated in Table 5 on page 383.

1364. SOLUTION. After applying formulae (12), (13) to the expression $w(z)$ given in the hint to the problem we obtain:

$w\left(\frac{1}{k}\right) - w(1) = 0 = C(k^{2b^2}K' - E')$, whence $b = 1/k\sqrt{(E'/K')}$. Then

$$w = C\left\{E(u) - \left(1 - \frac{E'}{K'}\right)u\right\} = C\left\{Z(u) + \frac{\pi u}{2KK'}\right\}.$$

(The correspondence between the w, z and u-planes is shown in Fig. 106).

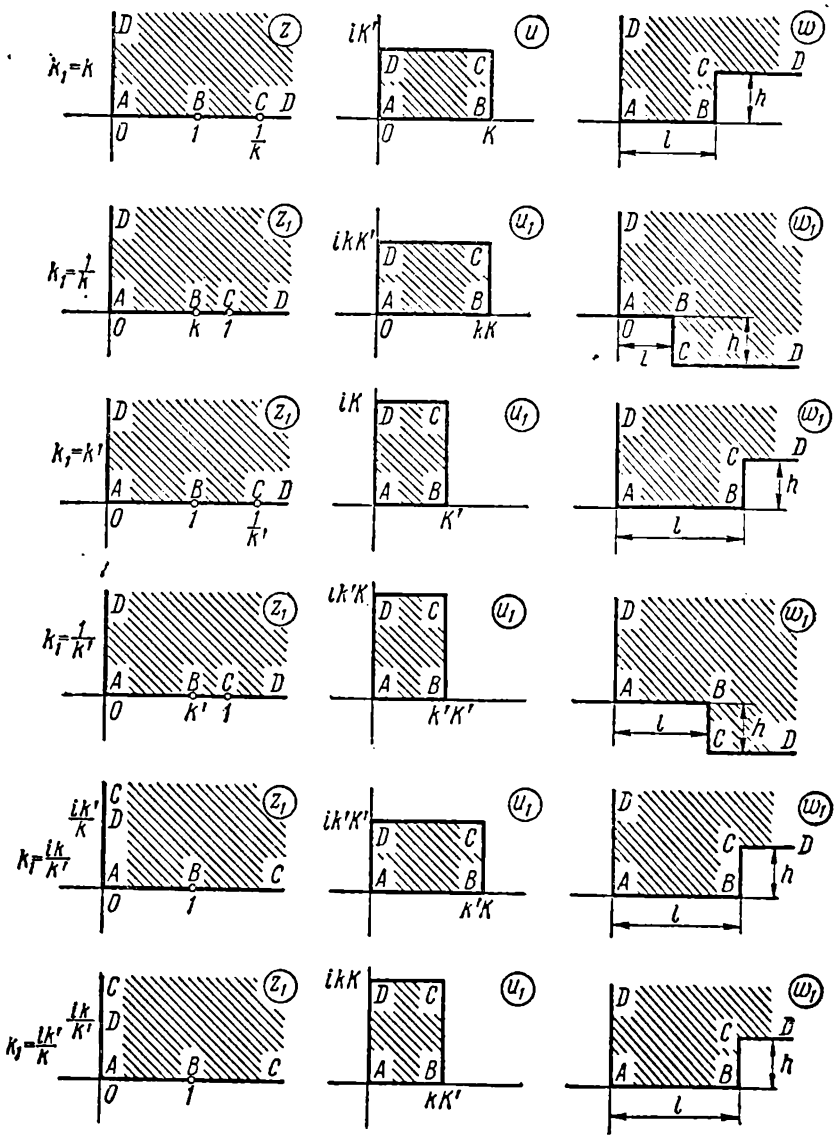

Fig. 105

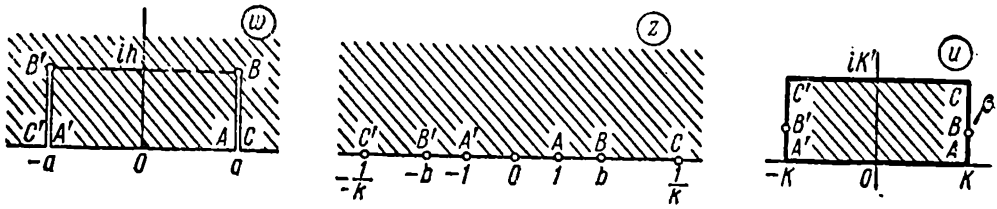

Fig. 106

TABLE 5

k_1	l	h
k	E	$K'-E'$
$\frac{1}{k}$	$\frac{1}{k}(E-k'^2K)$	$\frac{1}{k}(E'-k^2K')$
k'	E'	$K-E$
$\frac{1}{k'}$	$\frac{1}{k'}(E'-k^2K')$	$\frac{1}{k'}(E-k'^2K)$
$\frac{ik}{k'}$	$\frac{1}{k'}E$	$\frac{1}{k'}(K'-E')$
$\frac{ik'}{k}$	$\frac{1}{k}E'$	$\frac{1}{k}(K-E)$

For the determination of k and C we use the equalities

$$a = w(1) = C\left\{E - \left(1 - \frac{E'}{K'}\right)K\right\} = \frac{\pi C}{2K},$$

$$h = \frac{1}{i}[w(b) - w(1)] = C\left\{\int_0^{t_0} \sqrt{\left(\frac{1-k'^2t^2}{1-t^2}\right)}\,dt - \frac{E'}{K'}\int_0^{t_0} \frac{dt}{\sqrt{[(1-t^2)(1-k'^2t^2)]}}\right\},$$

where $t_0 = \frac{1}{k'}\sqrt{(1-k^2b^2)} = \sqrt{\frac{D'}{K'}}$ †.

1365. See Fig. 107. The position of the points B, B' on the u-plane is determined from the condition $dw/du = 0$, which leads to the equation

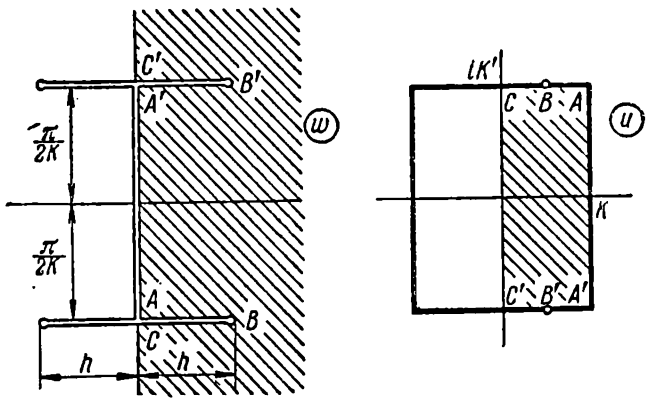

FIG. 107

† For details of the solution and graphs for the determination of the parameters see: A. BETZ, *Konforme Abbildung*, Berlin, 1948.

$\frac{\mathrm{cn}^2(u,k)}{\mathrm{sn}^2(u,k)}+\frac{E'}{K}=0$. After this since $u=\beta\pm iK'$ has been found we determine the value of h: $h=Z(\beta,k)$. If the parameter k is replaced by k', then the mapping reduces to the mapping of problem **1364** for $C=1$ and $a=\pi/2K$ with the substitution $u_1=K'+iu$, $w_1=iw$.

1366. Cases (1), (2), (3) and (4) are represented in Fig. 108, 109 respectively. In all cases for comparison the mapping has been given onto the u-plane by means of the normal elliptic integral of the first kind (1). The continuation of the mapping of the first quadrant I of the z-plane by the symmetry principle leads in the w-plane to a strip with a rectangular cavity (see the domains I + II in Fig. 108, 1, 2), to a strip with a rectangular projection (see the domains I+II in Fig. 109, 1, 2) and also to other domains (some of them are shown in the diagrams). For the fundamental dimensions H, l, h which then arise see Table 6. Note that the second case reduces to the first and the fourth to the third by replacing z by t: $k^2z^2+k'^2t^2=1$. Then instead of ν, k in cases (2) and (4) there occur the values $\nu'=-\frac{\nu}{k^2+\nu}k'^2$, k' and the corresponding w-figures are obtained from the w-figures for the cases (1) and (3) by means of entire linear transformations with coefficients of magnification $\frac{(\text{length})_2}{(\text{length})_1}=-\frac{k^2+\nu}{k^2}$, $\frac{(\text{length})_4}{(\text{length})_3}=\frac{k^2+\nu}{k^2}$ (the subscripts indicate the cases). The relations between the u and w-planes for the cases $\nu=-1$ and $\nu=-k^2$ are given in Fig. 110. Using Table 6 on page 387 and the hint to problem **1365**, we obtain for $\nu=-1$:

$$w=\frac{1}{k'^2}\left[k'^2u-E(u)+\frac{\mathrm{sn}\,u\,\mathrm{dn}\,u}{\mathrm{cn}\,u}\right]=-\frac{i}{k'}E\left(ik'u,\frac{1}{k'}\right),$$

$$l=\frac{1}{k'^2}(E-k'^2K),\quad h=\frac{1}{k'^2}(E'-k^2K')$$

and for $\nu=-k^2$:

$$w=\frac{1}{k'^2}\left[E(u)-k^2\frac{\mathrm{sn}\,u\,\mathrm{cn}\,u}{\mathrm{dn}\,u}\right]=\frac{1}{k'}E\left(k'u,\frac{ik}{k'}\right),\quad l=\frac{1}{k'^2}(K'-E'),\quad H=\frac{1}{k'^2}E.$$

1367. Solution. It follows from the condition $\Delta>0$ that e_1, e_2, e_3 are real and different and $g_2>0$. Let us consider that $e_1>e_2>e_3$. The upper half-plane Im $z>0$ is mapped onto the rectangle with vertices 0, ω, $\omega-\omega'$, $-\omega'$ (it is assumed that Im $\frac{\omega'}{\omega}>0$), corresponding to the points ∞, e_1, e_2, e_3. The median line of the rectangle corresponds to two semicircles (Fig. 111): the first with centre at the point e_3, with respect to which e_2 and e_1 are symmetrical (consequently, its radius is $\sqrt{[(e_1-e_3)(e_2-e_3)]}$, and the second with centre at the point e_1, with respect to which e_2, e_3 are symmetrical (its radius is $\sqrt{[(e_1-e_2)(e_1-e_3)]}$. Continuing the mapping of $\mathfrak{p}(w)$ by the symmetry principle we arrive at the half-periods ω, ω' of this function:

$$\omega=\int_{e_1}^{\infty}\frac{dx}{\sqrt{[(x-e_1)(x-e_2)(x-e_3)]}},$$

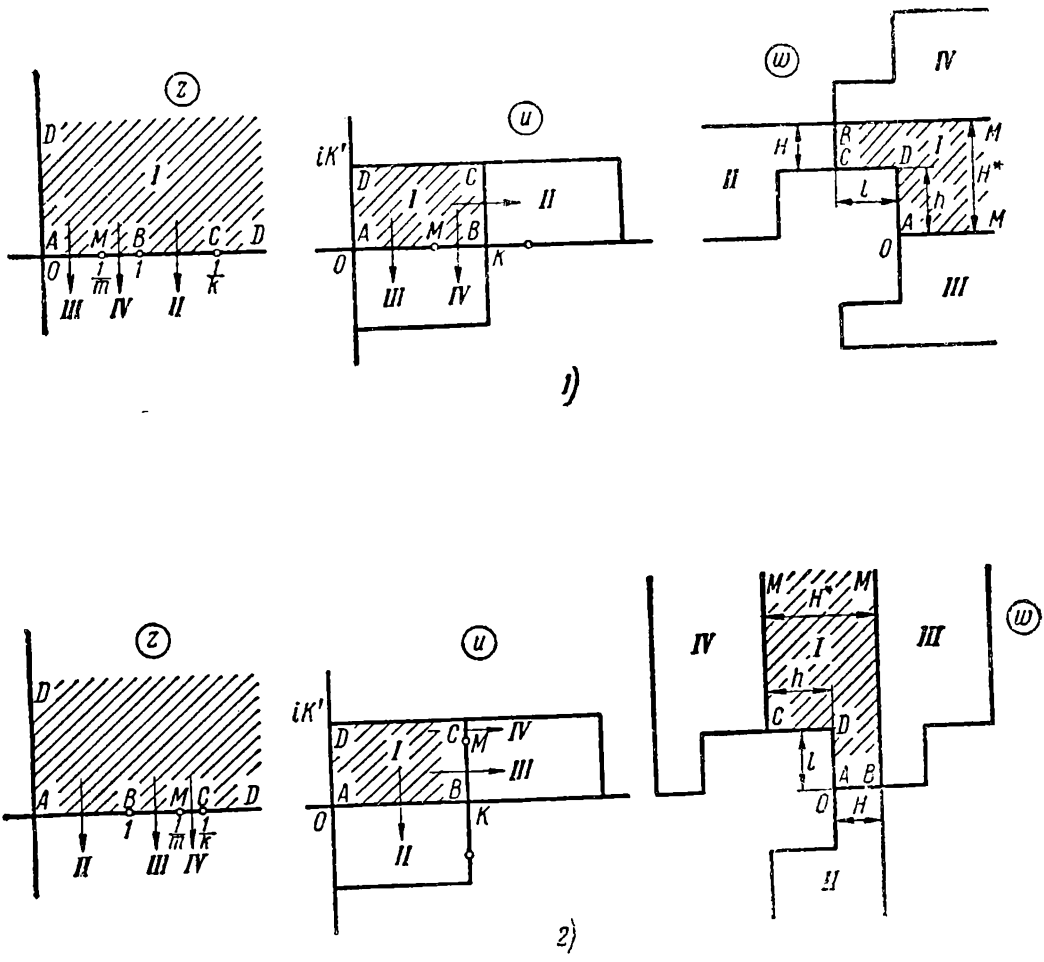
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Fig. 108

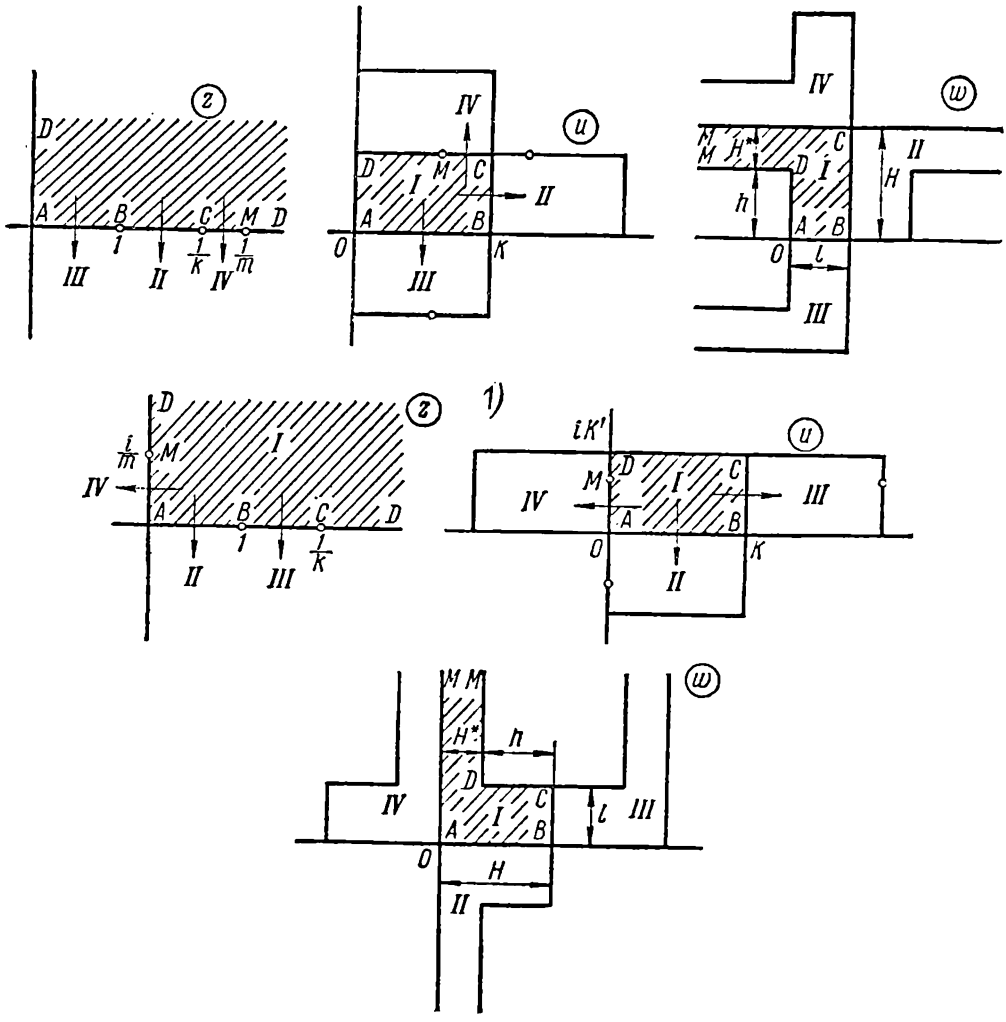

FIG. 109

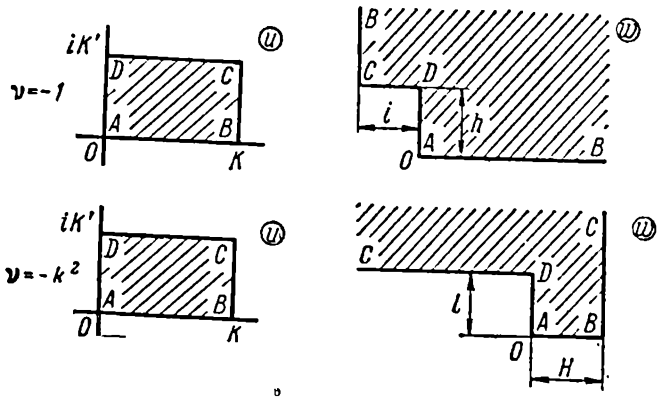

FIG. 110

TABLE 6

ν	H^*	H	l	h
$\nu < -1$	$\frac{\pi}{2}\sqrt{\frac{\lvert\nu\rvert}{\lvert\nu+1\rvert\,\lvert\nu+k^2\rvert}}$	$-\frac{k^2}{k^2+\nu}\Pi_1(\nu', k')$	$-\Pi_1(\nu, k)$	H^*-H
$-1 < \nu < -k^2$	$\frac{\pi}{2}\sqrt{\frac{\lvert\nu\rvert}{\lvert\nu+1\rvert\,\lvert\nu+k^2\rvert}}$	$\Pi_1(\nu, k)$	$\frac{k^2}{k^2+\nu}\Pi_1(\nu'', k')$	H^*-H
$-k^2 < \nu < 0$	$\frac{\pi}{2}\sqrt{\frac{\lvert\nu\rvert}{\lvert\nu+1\rvert\,\lvert\nu+k^2\rvert}}$	$\frac{k^2}{k^2+\nu}\Pi_1(\nu', k')$	$\Pi_1(\nu, k)$	$H-H^*$
$\nu > 0$	$\frac{\pi}{2}\sqrt{\frac{\lvert\nu\rvert}{\lvert\nu+1\rvert\,\lvert\nu+k^2\rvert}}$	$\Pi_1(\nu, k)$	$\frac{k^2}{k^2+\nu}\Pi_1(\nu', k')$	$H = H^*$
$\nu = -1$	—	—	$\frac{1}{k'^2}(E-k'^2 K)$	$\frac{1}{k'^2}(E'-k^2 K')$
$\nu = -k^2$	—	$\frac{1}{k'^2}E$	$\frac{1}{k'^2}(K'-E')$	—

REMARK. In the column "l" the integrals are understood in the sense of the principal value.

The quantity $\nu' = -\frac{\nu}{k^2+\nu}k'^2$.

$$\omega' = i \int_{-\infty}^{e_3} \frac{dx}{\sqrt{[(e_1-x)\ (e_2-x)\ (e_3-x)]}}\ .$$

From the consideration of Fig. 111 we arrive at the relations:

$$\wp(w) = e_3 + \frac{e_1 - e_3}{\operatorname{sn}^2\left(\frac{K}{\omega}\, w,\ k\right)},\ k^2 = \frac{e_2 - e_3}{e_1 - e_3}\ .$$

If $g_3 > 0$, then $e_3 < e_2 < 0 < e_1$ and $\omega < |\omega'|$ (for $k < k'$ and consequently $K < K'$); if $g_3 < 0$, then $e_3 < 0 < e_2 < e_1$, and consequently $\omega > |\omega'|$.

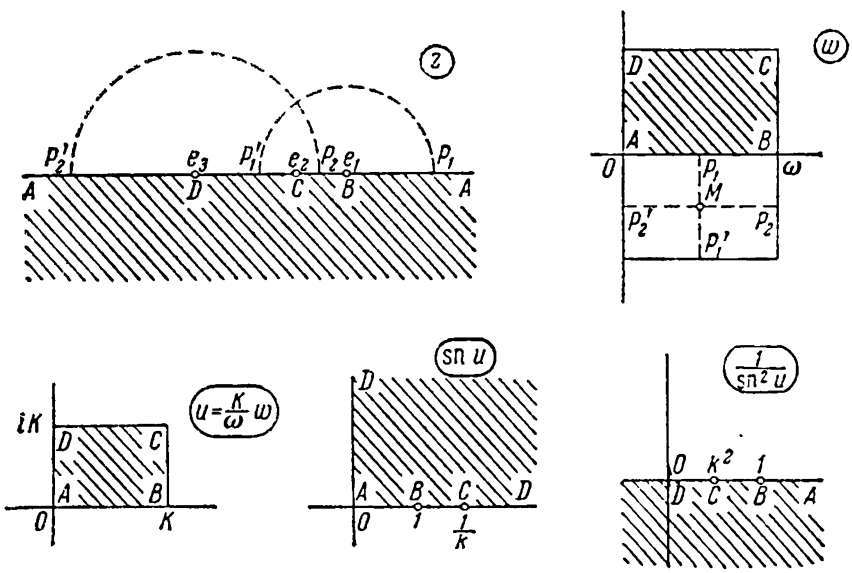

Fig. 111

If $g_3 = 0$, then $e_2 = 0$, $e_3 = -e_1$ and $\omega = |\omega'|$. In this case the mapping is also symmetrical with respect to the vertical axis. To the whole z-plane with the cuts $(-\infty, e_3]$, $[e_1, \infty)$, $[0, i\infty)$ there corresponds the triangle $(0,\ 2\omega,\ 2\omega')$, which forms half of the period parallelogram (this is now a square) (Fig. 112).

Let us also note that in the case of arbitrary e_1, e_2, e_3 $(e_1+e_2+e_3 = 0)$ half the period parallelogram corresponds to the z-plane with cuts, in general curvilinear, originating at e_1, e_2, e_3 and going off to ∞ (see the diagram in Fig. 113).

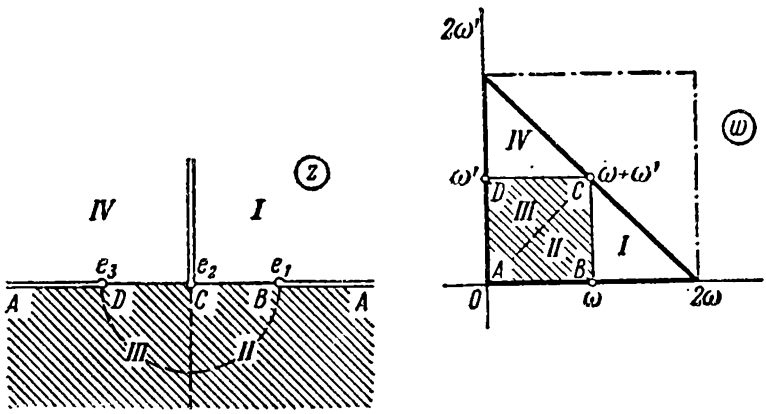

FIG. 112

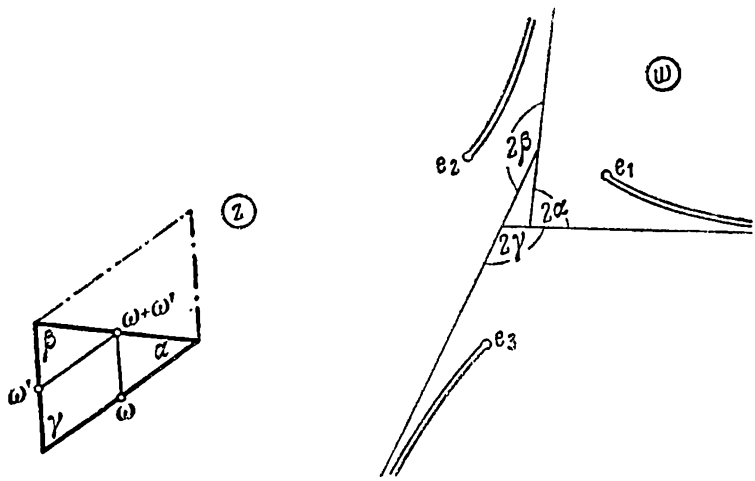

FIG. 113

1368. SOLUTION. The basic mapping is shown in Fig. 114. It is obtained by means of the symmetry principle from the mapping of the semicircle II. Noting that $dw = -\dfrac{dz}{\sqrt{[(z-e_1)(z-e_2)(z-e_3)]}}$ and taking $\arg(-1) = \pm\pi$

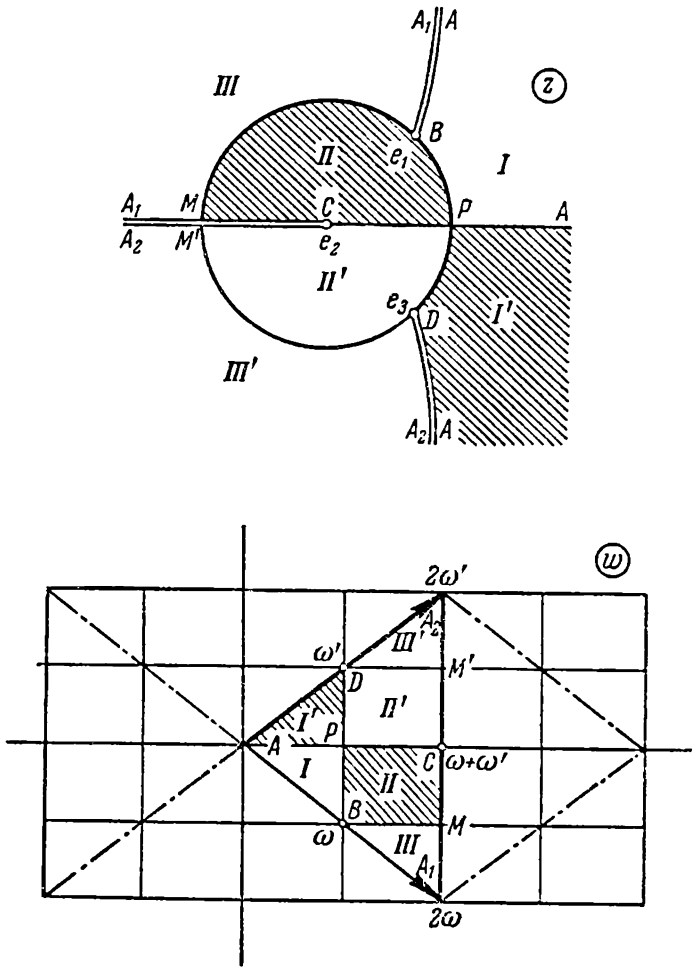

FIG. 114

we have $\arg dw = \pm\pi + \arg dz - \dfrac{1}{2}\sum\limits_{k=1}^{3} \arg(z-e_k)$, whence it follows that $\arg dw$ has the values $-\dfrac{\pi}{2}$, 0, $\dfrac{\pi}{2}$, $-\pi$ respectively, on the sides of the "quadrilateral" $PBMC$, which leads to the mapping shown in Fig. 114.

Thus, for example, on the arc $\overset{\smile}{PB}$ we have (Fig. 115)

$$\arg dw = -\pi + \left(\alpha_2 + \frac{\pi}{2}\right) - \frac{1}{2}(-\alpha_1+\alpha_2+\alpha_3) = -\frac{\pi}{2} + \frac{\alpha_1+\alpha_2-\alpha_3}{2} = -\frac{\pi}{2}$$

and so on. For the determination of the complex conjugates of the half-

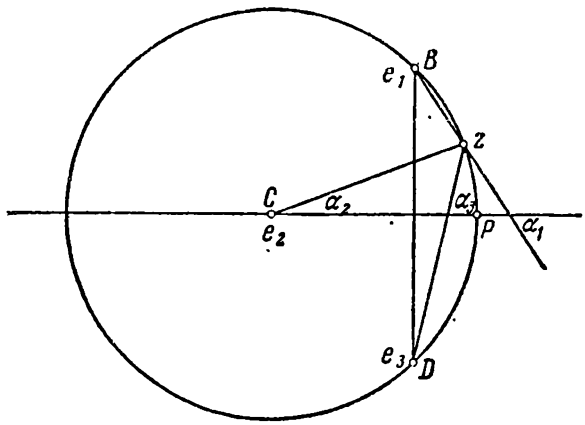

FIG. 115

periods ω, ω' of the function $\mathfrak{p}(w)$ we have:

$$\omega+\omega' = \int_{e_3}^{\infty} \frac{dx}{\sqrt{[(x-e_1)\ (x-e_2)\ (x-e_3)]}},$$

$$\omega-\omega' = \frac{i}{2}\int_{-\infty}^{e_3} \frac{dx}{\sqrt{[(e_1-x)\ (e_2-x)\ (e_3-x)]}}.$$

If $g_2 = 0$, then e_1, e_2, e_3 are the vertices of an equilateral triangle and the period parallelogram has the form of a rhomb with an angle of 60° at zero; if $g_3 < 0$ (then $e_2 < 0$), and an angle of 120° at zero if $g_3 > 0$ (then $e_2 > 0$). In both cases half the period parallelogram corresponds to the z-plane with symmetrical radial cuts going from the points e_1, e_2, e_3 (see Fig. 116, 117).

1369. (1) $w = -\left[\frac{\mathfrak{p}(z)}{e_1}\right]^2$, the half-periods are ω, $i\omega$;

(2) $w = \mathfrak{p}'(z)/2|e_2|^{3/2}$, the half-periods are $\omega = he^{-i\pi/6}$, $\omega' = he^{i\pi/6}$, where $h = a\sqrt{3}/2$;

(3) $w = \mathfrak{p}'^2(z)/4e_2^3$, the half-periods are the same as in part (2).

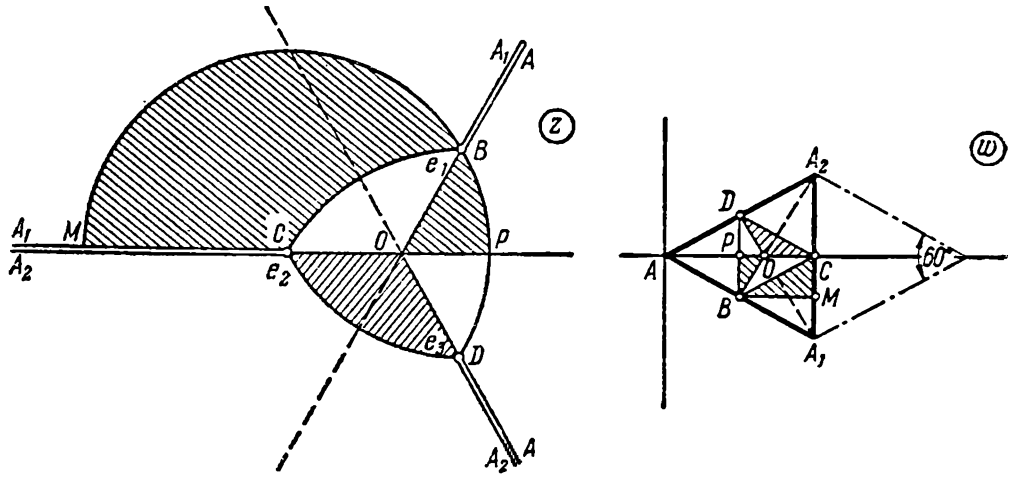

FIG. 116

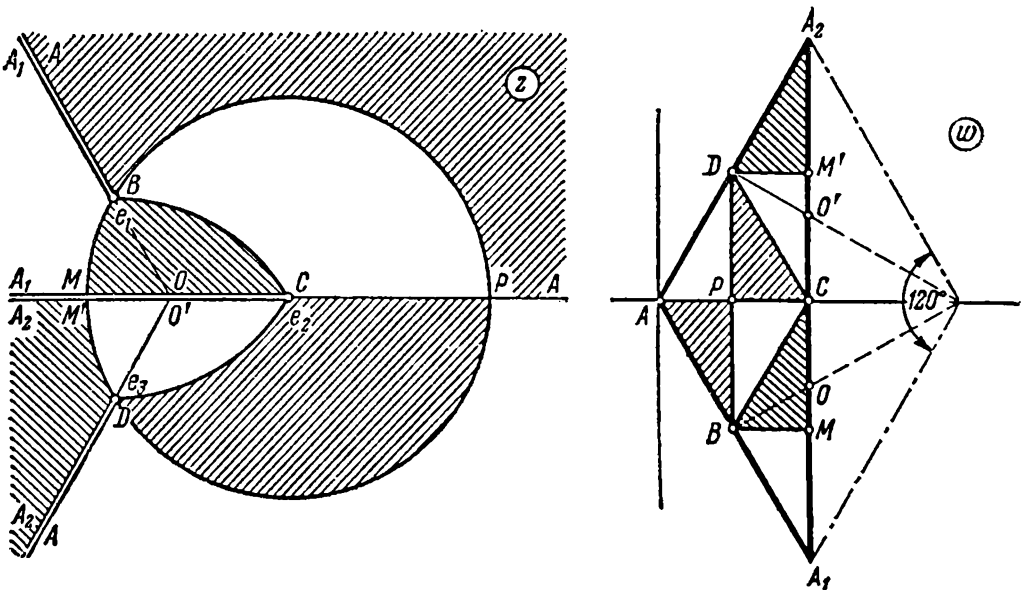

FIG. 117

1370. (1) $z = \operatorname{sn}(u, k)$, $w = e^{\pi u/K'}$, $\log \mu = 2\pi \frac{K'}{K}$; $|\operatorname{Re} u| < K$, $|\operatorname{Im} u| < K'$;

(2) the mapping function is the same as in part (1) except that $0 < \operatorname{Re} u < K$, $|\operatorname{Im} u| < K'$, $\log \mu = \pi K'/K$;

(3) reduces to part (1) by means of a linear transformation; then we have $k = \frac{\sqrt{\lambda} - \sqrt{(\lambda - 1)}}{\sqrt{\lambda} + \sqrt{(\lambda - 1)}}$ where $\lambda = (a, b, c, d)$ is the cross-ratio of the given points;

(4) reduces to part (3) by means of the mapping $t = \sqrt{(z^2 + h^2)}$;

(5) $z = \operatorname{sn}(u, k)$, $w = ie^{i\pi u/2K}$, $\log \mu = \frac{\pi}{2} \frac{K'}{K}$; $-3K < \operatorname{Re} u < K$, $0 < \operatorname{Im} u < K'$;

(6) reduces to part (5) by means of the mapping $t = \sqrt{(1 + z^2)}$; here $k = \cos \alpha$;

(7) the mapping function is the same as in part (5) except that in one case $0 < \operatorname{Im} u < \frac{1}{2} K'$, and in the other $\frac{1}{2} K' < \operatorname{Im} u < K'$; $\log \mu = \frac{\pi}{4} \frac{K'}{K}$;

(8) $z = k \operatorname{sn}^2(u, k)$, $w = e^{-i\pi u/K}$, $\log \mu = \pi \frac{K'}{K}$; $|\operatorname{Re} u| < K$, $0 < \operatorname{Im} u < K'$;

(9) the mapping function is the same as in part (8) except that in one case $0 < \operatorname{Im} u < \frac{1}{2} K'$ and in the other case $\frac{1}{2} K' < \operatorname{Im} u < K'$; $\log \mu = \frac{\pi}{2} \frac{K'}{K}$;

(10) is reduced to part (8) by means of the transformation $t = (z+1)/\sqrt{(1+\varrho)}$; here $k = 1/\sqrt{(1+\varrho)}$;

(11) is reduced to part (7) by means of the mapping $t = (1-z)/(1+z)$; here $k = \sin \frac{\alpha}{2}$;

(12) is reduced to part (2) by means of the mapping $t = -i \sin z$; here $k = \sinh H_1 / \sinh H_2$;

(13) is reduced to part (7) by the mapping $t = \sin z$; here $k = \cos \tanh H$;

(14) is reduced to part (5) by means of the mapping $t = \sin z$ and a subsequent linear transformation; here $k = \frac{\sqrt{\lambda} - \sqrt{(\lambda - 1)}}{\sqrt{\lambda} + \sqrt{(\lambda - 1)}}$, where $\lambda = \frac{(1 + \sin \beta)(1 - \sin \alpha)}{2(\sin \beta - \sin \alpha)}$;

(15) $z = C\left[Z(u) + \frac{\operatorname{dn} u \operatorname{cn} u}{\operatorname{sn} u}\right]$, $w = e^{-iu\pi/K}$, $\log \mu = 2\pi \frac{K'}{K}$,

where k is determined from the equations $\frac{\pi}{2K} : Z(\beta) = \tan \alpha$, $\operatorname{dn}^2 \beta = E/K$ (see problems **1364** and **1365**).

1371. $t = (1 - \operatorname{cn} u)/\operatorname{sn} u$. The vertices $K \pm iK'$, $-K \pm iK'$ pass into the points $e^{\pm i\alpha}$, $-e^{\pm i\alpha}$, where $\cos \alpha = k$.

1372. This reduces to problem **1371** by the mapping $u = \frac{2K}{\pi} \sin^{-1} z$. Here

k is determined from the equations $q = e^{-\pi \frac{K'}{K}} = \frac{1}{(a+b)^2}$, where $b = \sqrt{(a^2-1)}$.

1373. $t = \frac{1 - \operatorname{dn} u}{k \operatorname{sn} u}$, where $u = \frac{2K}{\pi} \sin^{-1} z$, $|\operatorname{Re} u) < K$, $|\operatorname{Im} u| < K'$. The parameter k has the same value as in the answer to problem **1372**. The foci $z = \pm 1$ pass into the points $\pm k/(1+k')$.

1374. (1) $z = C\left[Z(u) + \frac{\operatorname{dn} u \operatorname{cn} u}{\operatorname{sn} u}\right]$, $t = \frac{1 + \operatorname{cn} u}{\operatorname{sn} u}$; k is determined as in problem **1370**, (15). See also problems **1365** and **1371**;

(2) $z = C\left[Z(u) + \frac{\pi u}{2KK'}\right]$, $t = (1+\operatorname{cn} u)/\operatorname{sn} u$; for the determination of k see the footnote to the answer to problem **1364**;

(3) $z = \frac{a}{lk} E\left(ku, \frac{1}{k}\right) = \frac{a}{lk}\left[E(u, k) - i(E' - k^2K') - k'^2 u + \frac{\operatorname{cn}(u, k) \operatorname{dn}(u, k)}{\operatorname{sn}(u,k)}\right]$, $t = \frac{1 + \operatorname{cn} u}{\operatorname{sn} u}$, where k is determined from the equation $\frac{E' - k^2K'}{E - k'^2K} = \frac{b}{a}$ (see problem **1363** and Table 6).

CHAPTER XI

1376. Steady motion with velocity $V = \alpha - i\beta$. At ∞ there is a doublet with moment $p = 2\pi c$. The stream lines are $\beta x + \alpha y = C$; the equipotential lines are $\alpha x - \beta y = C$.

1377. At the point $z = 0$ there is a critical point (a stagnation point), at ∞ there is a multiplet of order $2n$ (also a stagnation point); $r^n \cos n\phi = C$ are equipotential lines, $r^n \sin n\phi = C$ are stream lines ($z = re^{i\phi}$); $V = n\bar{z}^{n-1}$.

1378. At the points $z = 0$ and $z = \infty$ there are vortex sources: $(0; Q, \Gamma)$, $(\infty; -Q, -\Gamma)$; $\log r = -\frac{\Gamma}{Q}\phi + C$ are equipotential lines; $\log r = \frac{Q}{\Gamma}\phi + C$ are stream lines. Both families of lines are logarithmic spirals; in the cases $\Gamma = 0$ or $Q = 0$ one of these families are circles $r = C$, the other are rays $\phi = C$. The velocity is $V = \frac{|\Gamma + iQ|}{2\pi r} e^{i\left(\phi - \tan^{-1}\frac{Q}{\Gamma} + \frac{\pi}{2}\right)}$ $(z = re^{i\phi})$.

1379. At the points a, b there are vortex sources: $(a; Q, \Gamma)$, $(b; -Q; \Gamma)$; the lines of the field are logarithmic spirals about the points a and b (Fig. 118); $\log \varrho = -\frac{\Gamma}{Q}\theta + C$ are equipotential lines; $\log \varrho = \frac{Q}{\Gamma}\theta + C$ are stream lines $\left(\frac{z-a}{z-b} = \varrho e^{i\theta}\right)$. The velocity is $V = \frac{\Gamma - iQ}{-2\pi i} \frac{\bar{a} - \bar{b}}{(\bar{z} - \bar{a})(\bar{z} - \bar{b})}$.

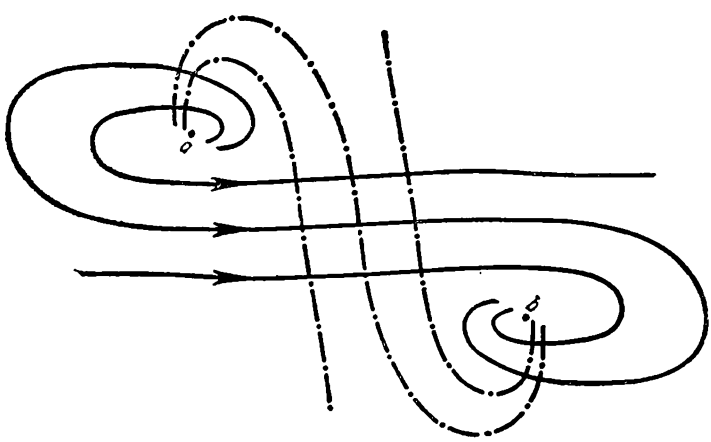

FIG. 118

1380. At the point $z = 0$ there is a doublet with moment $p = 2\pi$; $r = C \cos\phi$ are equipotential lines; $r = C \sin\phi$ are stream lines. $V = -e^{2i\phi}/r^2$, $V_\infty = 0$. At the points $2 \pm i$ the velocity equals $-(3 \pm 4i)/25$.

1381. (1) and (2). At the points 0 and ∞ there are doublets with moments $\pm 2\pi R^2$ and 2π (the upper sign refers to part (1), the lower to part (2)). The field lines are cubic curves: $x \pm R^2 x/(x^2 + y^2) = C$ are equipotential lines,

$y \mp Ry/(x^2+y^2) = C$ are stream lines. $V = 1 \mp \frac{R^2}{r^2} e^{2i\phi}$, $V_\infty = 1$. The points $z = \pm R$ for part (1) and $z = \pm Ri$ for part (2) are critical points (see Fig. 119 and 120).

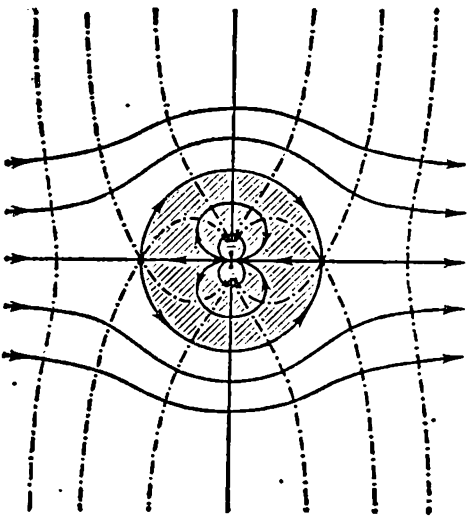

FIG. 119

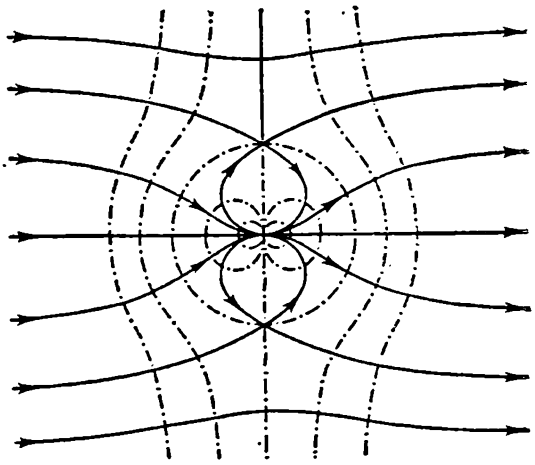

FIG. 120

1382. There is a quadruplet at the point $z = 0$; $r^2 = C \cos 2\phi$ are the equipotential lines; $r^2 = C \sin 2\phi$ are the stream lines. $V = -\frac{2}{r^3} e^{3i\phi}$. $V_\infty = 0$ (Fig. 121)†.

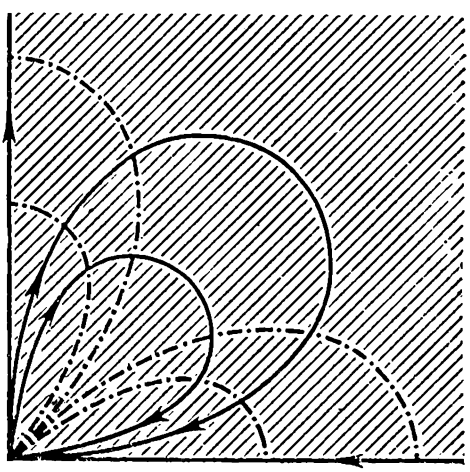

FIG. 121

† Here and in what follows where the flow is symmetrical with respect to the coordinate axes, it is shown only in the first quadrant.

1383. At the points $\pm a$ there are sources of strength 2π, at ∞ there is a source of strength -2π. The equipotential lines are the lemniscates $|z^2 - a^2| = C$; the stream lines are the hyperbolas $x^2 - \frac{2}{C}xy - y^2 = a^2$ with centre at the coordinate origin which pass through the points $\pm a$. $\mathbf{V} = \frac{2\bar{z}}{\bar{z}^2 - a^2}$, $\mathbf{V}_{\pm ia} = \pm \frac{i}{a}$; the coordinate origin is a critical point (Fig. 122).

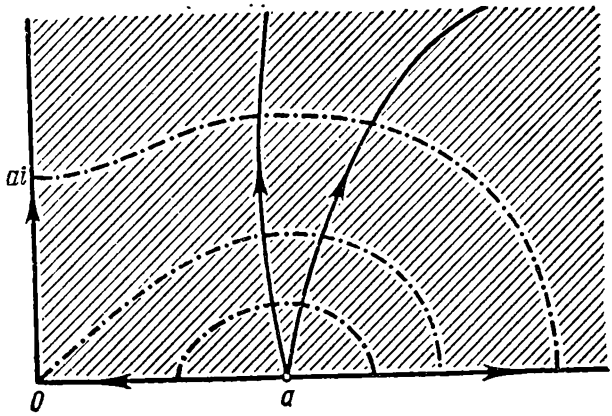

FIG. 122

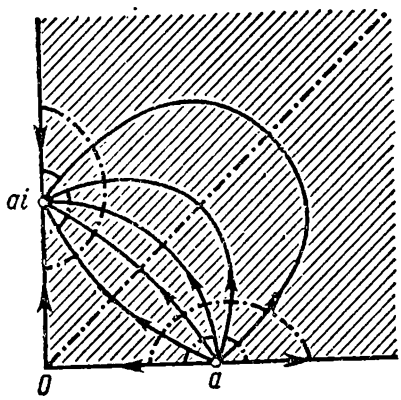

FIG. 123

1384. At the points $\pm a$, $\pm ai$ there are sources: $(\pm a;\ 2\pi)$, $(\pm ai;\ -2\pi)$; $r^4 + Ca^2r^2 \cos 2\varphi + a^4 = 0$ $(|c| > 2)$ are the equipotential lines (the straight lines $y = \pm x$ also belong to them); $r^4 + Ca^2r^2 \sin 2\phi - a^4 = 0$ are the stream lines (the coordinate axes also belong to them). $\mathbf{V} = \frac{4a\bar{z}^2}{\bar{z}^4 - a^4}$, $\mathbf{V}_\infty = 0$, the point $z = 0$ is a critical point (Fig. 123).

1385. There are sources at the points ± 1, 0 and at ∞; $(\pm\ 1;\ Q)$, $(0; -Q)$,

$(\infty;\ -Q)$; $r^2+\frac{1}{r^2}=C+2\cos 2\phi$ $(C>0)$ are the equipotential lines (for large values of C the curves are close to the circles $r=\sqrt{C}$ and $r=1/\sqrt{C}$); $r=\sqrt{\left(\frac{C+\tan\phi}{C-\tan\phi}\right)}$ are the stream lines (they include also the coordinate axes and the circle $r=1$). $V=\frac{Q}{2\pi}\left[\overline{\frac{z^2+1}{z(z^2-1)}}\right]$, $V_\infty=0$. The points $\pm i$ are critical points (Fig. 124).

FIG. 124

1386. At the points $\pm i$, 0 there are the sources: $(\pm i;\ 2\pi)$; $(0;\ -4\pi)$; $Cr^4-2r^2\cos 2\phi-1=0$ $(C>-1)$ are equipotential lines (for $C=0$ a hyperbola $y^2-x^2=\frac{1}{2}$); $r=\sqrt{(C\sin 2\phi-\cos 2\phi)}$ are stream lines (for $C=0$ a lemniscate of Bernoulli), to which the coordinate axis also belongs. $V=-2/[\overline{z(z^2+1)}]$, $V_\infty=0$ (Fig. 125).

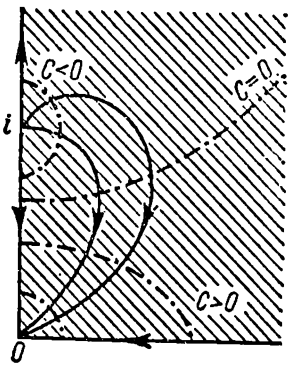

FIG. 125

1387. At the points $\pm(1\pm i)/\sqrt{2}$ there are sources of strength 2π, at the point $z=0$ a source of strength -4π, at ∞ a source of strength -4π; $r^4+\frac{1}{r^4}=C-2\cos 4\phi$ $(C>0)$ are equipotential lines (for $C<4$ the curves separate into four components, for $C>4$ into two; for large values of C they are "almost" the circles $r=\sqrt[4]{C}$ and $r=1/\sqrt[4]{C}$); $\tan 2\phi=(Cr^4-1)/(r^4+1)$ are stream lines (they also include the coordinate axes, the bisectors of the coordinate angles and the circle $r=1$). $V=2\left[\frac{z^4-1}{z(z^4+1)}\right]$, $V_\infty=0$. The points ± 1, $\pm i$ are critical points (Fig. 126).

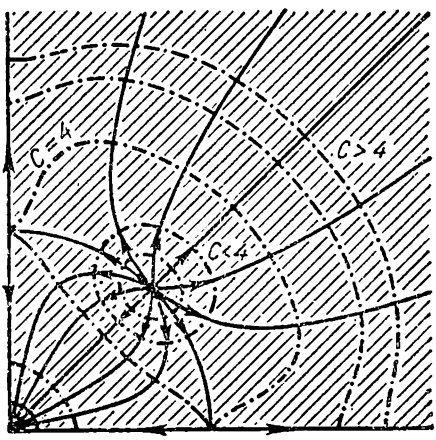

Fig. 126

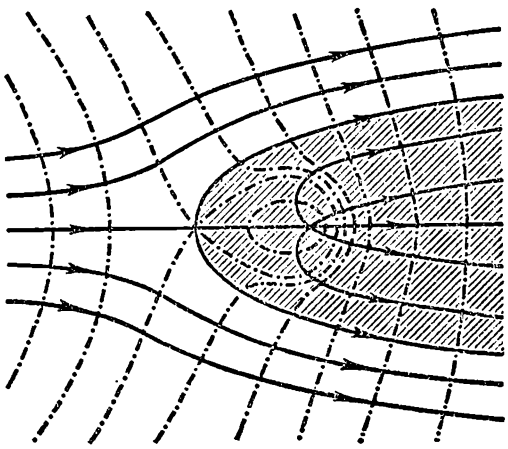

Fig. 127

1388. At the point $z=0$ there is the source $(0;\ Q)$, at the point $z=\infty$ a doublet and the source $(\infty;\ -Q)$; $y^2=e^{\frac{4\pi}{Q}(C-ax)}-x^2\left(\cos^{-1}\phi+\frac{Q}{2\pi}\log r=C\right)$ are the equipotential lines; $r=\left(C-\frac{Q}{2\pi}\phi\right)\Big/a\sin\phi$ are the stream lines; the stream lines have the horizontal asymptotes: $y_{x\to\infty}\to\frac{C}{a}$, $y_{x\to-\infty}\to\frac{C-\frac{Q}{2}}{a}$, $V=a+\frac{Qe^{i\phi}}{2\pi r}$, $V_\infty=a$; $z=-\frac{Q}{2\pi a}$ is a critical point (Fig. 127).

1389. At the point $z=0$ there is the vortex $(0;\ \Gamma)$, at ∞ the doublet and vortex $(\infty;\ -\Gamma)$; $r=(C-\Gamma\phi/2\pi)/a\cos\phi$ are equipotential lines;

$x^2 = e^{\frac{4\pi}{\Gamma}(ay-C)} - y^2\left(\sin^{-1}\phi - \frac{\Gamma \log r}{2\pi} = C\right)$ are stream lines; $V = a + \frac{\Gamma}{2\pi r} \times$ $\times e^{i\left(\phi+\frac{\pi}{2}\right)}$, $V_\infty = a$; the critical point is $z = \frac{\Gamma i}{2\pi a}$ (Fig. 128).

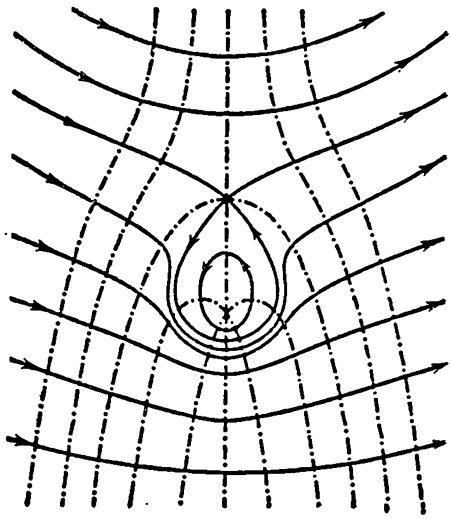

FIG. 128

1390. The fluid stream lines the circle of radius R; $V_\infty = a$, the circulation is Γ; the critical points are determined by the equation $z_{kr} = \frac{1}{4\pi a}(\Gamma i \pm \sqrt{(16\pi^2 a^2 R^2 - \Gamma^2)})$. If $\Gamma < 4\pi aR$, then $|z_{kr}| = R$, that is, both stagnation points lie on the circle $|z| = R$; if $\Gamma = 4\pi aR$, then the stagnation points merge into one; if $\Gamma > 4\pi aR$, then $|z_{kr}| > R$ (the second stagnation point is in this case inside the circle $|z| = R$). See, for example, [3, Chapter III, § 49].

1391. $w(z) = Ve^{-i\alpha}z + \sum_{k=1}^{n} \frac{\Gamma_k + iQ_k}{2\pi i} \log(z - a_k)$. At infinity there is a doublet with moment $2\pi Ve^{-i\alpha}$ and a vortex source of strength $Q_\infty = \sum_{k=1}^{n} Q_k$ and intensity $\Gamma_\infty = -\sum_{k=1}^{n} \Gamma_k$.

1392. (1) No; (2) Yes; (3) Yes (for example, the flow $w = \frac{1}{z} + \frac{\Gamma}{2\pi i} \log z$ has a stream line originating from the coordinate origin).

1393. In a schlicht conformal mapping a vortex source is transformed into a vortex source of the same strength and intensity. A multiplet is transformed into a set of multiplets of the same order inclusively. A doublet is

transformed into a doublet with the following relations between the moments:

(1) $(a;\ p) \to (\alpha;\ pc_1)$, (2) $(\infty;\ p) \to (\alpha;\ pc_{-1})$, (3) $(a;\ p) \to \left(\infty;\ \frac{p}{c_{-1}}\right)$

(4) $(\infty;\ p) \to \left(\infty;\ \frac{p}{c_1}\right)$.

1394. In a conformal mapping onto n sheets a vortex source is transformed into a vortex source with the strength and intensity decreased by a factor equal to n.

1395. The law of the change in the vortex source is (a^* is the point symmetrical to the point a): $(a;\ Q,\ \Gamma) \to (a^*;\ Q,\ -\Gamma)$ in the case of a stream line and $(a;\ Q,\ \Gamma) \to (a^*;\ -Q,\ \Gamma)$ in the caso of an equipotential line. The law giving the change in a doublet is more complicated. In the case of a rectilinear stream line: $(a;\ p) \to (a^*;\ p')$, where the vectors p, p', are drawn through a and a^* respectively, symmetrical with respect to the stream line. In the case of a circular stream line $|z| = R$: $(a;\ p) \to \left(a^*;\ -\frac{R^2}{\bar{a}^2}\bar{p}\right)$, if $a \neq 0$ and $(a;\ p) \to \left(\infty;\ \frac{\bar{p}}{R^2}\right)$, if $a = 0$. In the case of rectilinear and circular equipotential lines using the same notation we have respectively: $(a;\ p) \to (a^*;\ -p')$; $(a;\ p) \to \left(a^*;\ \frac{R^2}{\bar{a}^2}\bar{p}\right)$, $(0;\ p) \to (\infty;\ -\bar{p}/R^2)$.

1396. (1) In all cases the singularities of the flow must be symmetrically situated with respect to the circle $|z| = R$ (see problem **1395**). In particular the axes of the doublets must be situated on this circle, they must be tangential to it. If vortices occur on $|z| = R$ they must also have doublets. Moreover, the total strength and intensity must be equal to zero, for which: $\sum Q_k + \frac{1}{2}\sum Q_i' = 0$, where Q_k are the strengths of the sources within $|z| = R$ and Q_i' are the strengths of the sources on $|z| = R$; $\sum \Gamma_j' = 0$ where the Γ_j' are the intensities of the vortices on $|z| = R$.

(2) The singularities of the flow must be symmetrically situated with respect to the circle $|z| = R$ (see problem **1375**). In particular, the axes of doublets situated on $|z| = R$ must be orthogonal to it. If there are sources on $|z| = R$ they must also have doublets. Moreover the sums of the strengths and of the intensities must be equal to zero, for which: $\sum \Gamma_k + \frac{1}{2}\sum \Gamma_i' = 0$, where the Γ_k are the intensities of the vortices inside $|z| = R$ and the Γ_i' are the intensities of the vortices on $|z| = R$; $\sum Q_j' = 0$, where the Q_j' are the strengths of the sources on $|z| = R$.

1397. (1) $w = Vz + c$ (c is a constant); (2) $w = \frac{\Gamma}{2\pi i}\log\frac{z-a}{z-\bar{a}} + c$;

(3) $w = \frac{Q}{2\pi}\log[(z-a)(z-\bar{a})] + c$ (at ∞ there is a source of strength $-2Q$);

(4) $w = \frac{p}{2\pi}\frac{1}{z-a} + \frac{\bar{p}}{2\pi}\frac{1}{z-\bar{a}} + c$;

(5) $w=\sum_{k=1}^{n}\left[\frac{\Gamma_k+iQ_k}{2\pi i}\log(z-a_k)+\frac{-\Gamma_k+iQ_k}{2\pi i}\log(z-\bar{a}_k)\right]$

$$+\frac{p}{2\pi}\frac{1}{z-a}+\frac{\bar{p}}{2\pi}\frac{1}{z-\bar{a}}+Vz+c$$

$\left(\text{at } \infty \text{ there is a source of strength } -\sum_{k=1}^{n} Q_k\right)$;

(6) $w=\frac{p}{2\pi z}+\frac{\Gamma+iQ}{2\pi i}\log z+c$, where $\operatorname{Im} p=0$. A flow with a vortex without a doublet is impossible.

1398. (1) $w=\frac{\Gamma}{2\pi i}\log\frac{z-a}{R^2-\bar{a}z}+c$;

(2) $w=\frac{p}{2\pi}\frac{1}{z-a}+\frac{p^*}{2\pi}\frac{1}{z-a^*}+c$, if $a\neq 0$ $\left(a^*=\frac{R^2}{\bar{a}},\ p^*=-\frac{R^2}{\bar{a}^2}\bar{p}\right)$,

and $w=\frac{p}{2\pi z}+\frac{p^*}{2\pi}z+c$, if $a=0$ $\left(p^*=\frac{\bar{p}}{R^2}\right)$,

1399. (1) $w=\sum_{k=1}^{n}\frac{Q_k}{2\pi}\log[(z-a_k)(R^2-\bar{a}_k z)]+c$, if $\sum_{k=1}^{n} Q_k=0$;

(2) $w=\sum_{k=1}^{n}\frac{Q_k}{2\pi}\log[(z-a_k)(R^2-\bar{a}_k z)]$

$$+\sum_{i=1}^{m}\frac{Q_i}{2\pi}\log(z-a_i')+c, \quad \text{if} \quad \sum_{k=1}^{n} Q_k+\frac{1}{2}\sum_{i=1}^{m} Q_i'=0.$$

1400. (1) $w=\frac{\Gamma}{2\pi i}\log\frac{z-a}{R^2-\bar{a}z}+c$;

(2) $w=\frac{p}{2\pi}\frac{1}{z-a}+\frac{p^*}{2\pi}\frac{1}{z-a^*}+c$ $\left(a^*=\frac{R^2}{\bar{a}},\ p^*=-\frac{R^2}{\bar{a}^2}p\right)$;

(3) $w=Ve^{-i\alpha}z+\frac{R^2Ve^{i\alpha}}{z}+c$;

(4) $w=Ve^{-i\alpha}z+\frac{R^2Ve^{i\alpha}}{z}+\frac{\Gamma}{2\pi i}\log z+c$.

1401. $w=\frac{\Gamma}{2\pi i}\log\frac{(z-ia)(az+i)}{(z+ia)(az-i)}+c$.

1402. $w=\frac{Q}{2\pi}\log(z^6+a^6)+c$. **1403.** $w=\frac{Q}{2\pi}\log(z^2-1)+c$.

1404. $w = \dfrac{Q}{2\pi} \log \dfrac{z^2-1}{z^2+1} + c.$ **1405.** $w = \dfrac{Q}{2\pi} \log \left(1 + \dfrac{4}{z^4}\right) + c.$

1406. $$w = \sum_{k=1}^{n} \left[\frac{\Gamma_k + iQ_k}{2\pi i} \log (z-a_k) + \frac{\Gamma_k - iQ_k}{2\pi i} \log (z-\bar{a}_k) \right] + \frac{p}{2\pi} \cdot \frac{1}{z-a} - \frac{\bar{p}}{2\pi} \cdot \frac{1}{z-\bar{a}} + V e^{-i\alpha} z + c.$$

The flow is possible if $\alpha = \pm \dfrac{\pi}{2}$ $\left(\text{if } \sum_{k=1}^{n} \Gamma_k \neq 0, \text{ then at } \infty \text{ there is a vortex of intensity } -2 \sum_{k=1}^{n} \Gamma_k\right)$.

1407. $$w = \sum_{k=1}^{n} \left[\frac{\Gamma_k + iQ_k}{2\pi i} \log (z-a_k) + \frac{\Gamma_k - iQ_k}{2\pi i} \log (R^2 - \bar{a}_k z) \right] + \frac{p}{2\pi} \cdot \frac{1}{z-a} + \frac{p^*}{2\pi} \cdot \frac{1}{z-a^*} + V e^{-i\alpha} z - \frac{R^2 V e^{i\alpha}}{z} + c$$

$$\left(a^* = \frac{R^2}{\bar{a}},\ p^* = \frac{R^2}{\bar{a}^2} \bar{p}\right).$$

The flow is possible if $\sum_{k=1}^{n} \Gamma_k = 0$.

1408. Let $t = f(z)$ map D conformally onto the unit disk $|t| < 1$. Then $w = \Phi[f(z)]$, where

$$\Phi(t) = \sum_{k=1}^{n} \left[\frac{\Gamma_k + iQ_k}{2\pi i} \log (t-t_k) + \frac{-\Gamma_k + iQ_k}{2\pi i} \log (1-\bar{t}_k t) \right] + c,$$

$t_k = f(a_k)$, with the indispensable condition $\sum_{k=1}^{n} Q_k = 0$.

1409. Let $t = f(z)$ map D conformally onto the domain $|t| > 1$ with the normalisation $f(\infty) = \infty$, $f'(\infty) > 0$. Then with the condition $\sum_{k=1}^{n} Q_k = 0$ we have $w = \Phi[f(z)]$, where

$$\Phi(t) = \frac{V e^{-i\alpha}}{f'(\infty)} t + \frac{V e^{i\alpha}}{f'(\infty).t} + \sum_{k=1}^{n} \left[\frac{\Gamma_k + iQ_k}{2\pi i} \log (t-t_k) + \frac{-\Gamma_k + iQ_k}{2\pi i} \log (1-\bar{t}_k t) \right] + c,\ t_k = f(a_k).$$

1410. In the notation of problem **1409**, $w = \Phi[f(z)]$, where

$$\Phi(t) = \frac{Ve^{-i\alpha}}{f'(\infty)}t + \frac{Ve^{i\alpha}}{f'(\infty).t} + \frac{\Gamma}{2\pi i}\log t + c.$$

For $\Gamma = 0$, $w(z)$ maps the exterior of C onto the exterior of the segment $\left[-\frac{2V}{f'(\infty)}, \frac{2V}{f'(\infty)}\right]$ of the real axis of the w-plane with the normalisation $w(\infty) = \infty, w'(\infty) = Ve^{-i\alpha}$.

1411. (1) $w(z) = \frac{V}{a-b}[(az-b\sqrt{(z^2-c^2)}\cos\alpha + i(bz-a\sqrt{(z^2-c^2)})\sin\alpha] + \text{const}$ $(V_\infty = Ve^{i\alpha}, c = \sqrt{(a^2-b^2)})$;

(2) $w(z) = \frac{V}{a-b}[(az-b\sqrt{(z^2-c^2)})\cos\alpha + i(bz-a\sqrt{(z^2-c^2)})\sin\alpha]$

$+ \frac{\Gamma}{2\pi i}\log(z+\sqrt{(z^2-c^2)}) + \text{const}.$

1412. (1) $w(z) = V(z\cos\alpha - i\sqrt{(z^2-c^2)}\sin\alpha) + \text{const}$ $(V_\infty = Ve^{i\alpha})$;

(2) $w(z = V(z\cos\alpha - i\sqrt{(z^2-c^2)}\sin\alpha) + \frac{\Gamma}{2\pi i}\log(z+\sqrt{(z^2-c^2)}) + \text{const}$,

where $\Gamma = -2\pi cV\sin\alpha$ (c is a point of departure).

1413. Let the Zhukovskii profile be obtained by the mapping $z = (\zeta+\zeta^{-1})/2$ of the circle $|\zeta-\zeta_0| = |1-\zeta_0| = R > 1$, $\zeta_0 = 1-Re^{-i\beta}$ $(0 \leqslant \beta < \pi/2)$. Then for circulation Γ and $V_\infty = Ve^{i\alpha}$;

$$w(z) = \frac{VR}{2}\left(\frac{z-\zeta_0+\sqrt{(z^2-1)}}{R}e^{-i\alpha} + \frac{Re^{i\alpha}}{z-\zeta_0+\sqrt{(z^2-1)}}\right) + \frac{\Gamma}{2\pi i}\log[z-\zeta_0+\sqrt{(z^2-1)}] + c,$$

where $\Gamma = -2\pi RV\sin(\alpha+\beta)$ (Γ is determined from the condition $w'(1) = 0$ corresponding to the Zhukovskii–Chaplygin postulate).

1414. $w(z) = \sqrt{[z-(p/2)]} + c$ is the streamlining of the parabola from outside; $w(z) = i\cosh\sqrt{(2z-p)/2}\sqrt{p}$ is the streamlining of the parabola from inside.

1415. $w(z) = \frac{1}{\sqrt{2}}[(z+\sqrt{(z^2-c^2)})^{\frac{\pi}{2\beta}}e^{-\frac{i\pi\alpha}{2\beta}} - (z-\sqrt{(z^2-c^2)})^{\frac{\pi}{2\beta}}e^{\frac{i\pi\alpha}{2\beta}}] + \text{const}$

is the streamlining of the right hand branch of the hyperbola from outside $(\tan\alpha = \frac{b}{a}, \beta = \pi-\alpha, c = \sqrt{(a^2+b^2)}$ and

$$w(z) = \frac{i}{\sqrt{2}}[(z+\sqrt{(z^2-c^2)})^{\frac{\pi}{2\alpha}} + (z-\sqrt{(z^2-c^2)})^{\frac{\pi}{2\alpha}}] + \text{const}$$

is the streamlining of the right hand branch of the hyperbola from inside.

1416. $w(z)$ is determined from the equation $z = e^{\pi w/v} + \pi w/v$ (the values of the stream function on the streamlined half-lines are taken as equal to $\pm v$).

1417. $w = \cosh^{-1} z = \operatorname{Log}[z+\sqrt{(z^2-1)}]$ (the values of the stream function on the streamlined half-lines are taken as equal to 0 and π).

1418. (1) A flow with period π; at the points $k\pi$ (k an integer) there are sources of strength Q; the points $\frac{\pi}{2}+k\pi$ are stagnation points. The velocity at infinity in a strip of periods is $V_\infty = V(x \pm i\infty) = \mp\frac{Q}{2\pi}i$. See Fig. 129 for the stream lines and equipotential lines.

(2) The same, only instead of sources at the points $k\pi$ there are vortices of strength Γ and $V(x \pm i\infty) = \mp\Gamma/2\pi$. In order to construct the field the stream lines and equipotential lines in Fig. 129 must be interchanged.

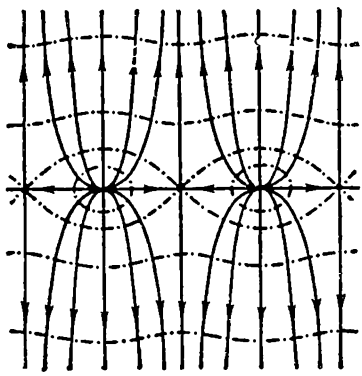

FIG. 129

1419. A flow with period π; at the points $k\pi$ there are doublets with moment p; the velocity is $V(x \pm i\infty) = 0$. For the stream lines see Fig. 130.

1420. If $V_1 = iV$, then the solution is possible and unique for $V_2 = i\left(V - \frac{2Q}{\pi}\right)$:

$$w = \frac{\Gamma+iQ}{2\pi i}\log\sin\frac{\pi(z-a)}{2\omega} + \frac{-\Gamma+iQ}{2\pi i}\log\sin\frac{\pi(z+\bar{a})}{2\omega} + i\left(V-\frac{Q}{\pi}\right)z+c.$$

1421. $$w = \frac{p}{2\pi}\cot\frac{\pi(z-a)}{2\omega} - \frac{\bar{p}}{2\pi}\cot\frac{\pi(z+\bar{a})}{2\omega} + iVz+c.$$

1422. Let $t = f(z)$ map S conformally onto the rectilinear strip S_t so that Ω_1, Ω_2 pass into points at infinity of S_t. If there exist non-zero derivatives $f'(\Omega_1), f'(\Omega_2)$, the velocities V_1, V_2 touch the boundary of S at infinity and one of them is given arbitrarily, then for S_t the problem reduces to problems **1420, 1421**; the solution exists and is unique.

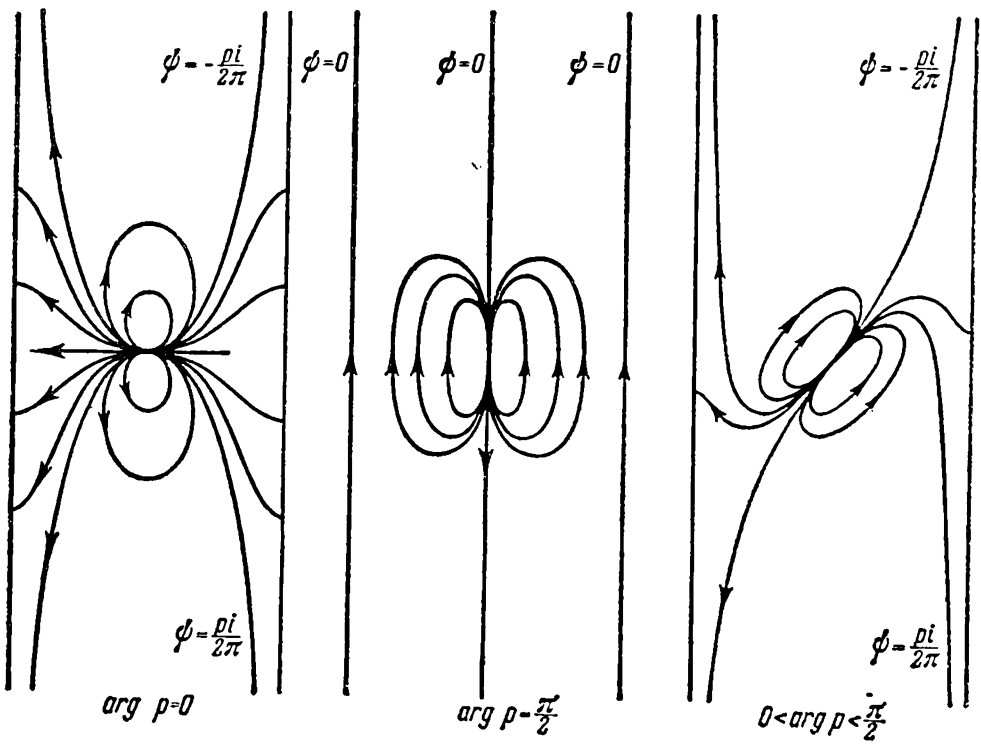
$\psi = -\frac{pi}{2\pi}$
$\psi = 0$
$\psi = \frac{pi}{2\pi}$
arg $p = 0$
$\psi = 0$
$\psi = 0$
arg $p = \frac{\pi}{2}$
$\psi = -\frac{pi}{2\pi}$
$\psi = \frac{pi}{2\pi}$
$0 < \arg p < \frac{\pi}{2}$

FIG. 130

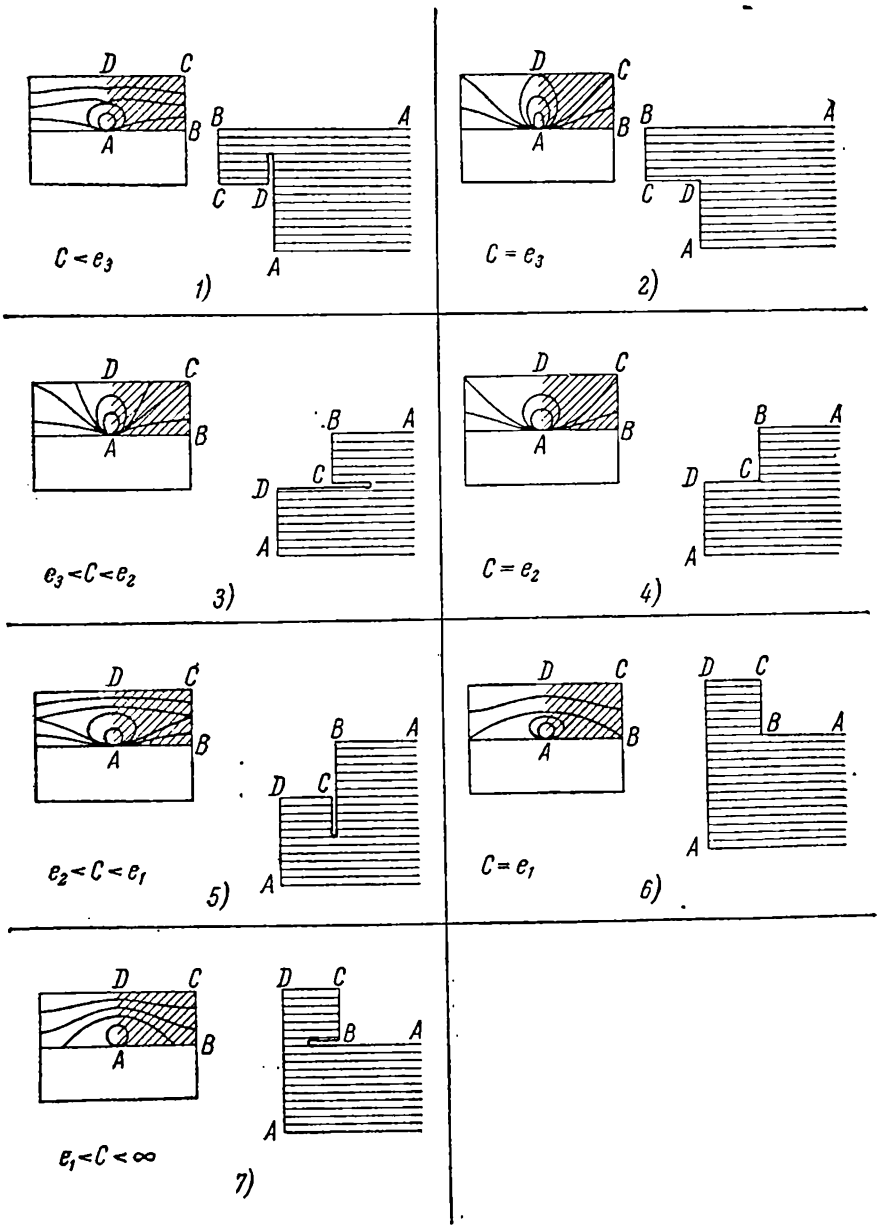

FIG. 131

1423. (1) It is necessary and sufficient that the numbers M and C should be real. In this case the curves $\operatorname{Re} u = \pm\,\omega$ are equipotential lines. In Fig. 131 the mapping $t = f(u) = \zeta(u) + Cu$ is shown for various real C. Figure 131 corresponds to the case $\omega > |\omega'|$. By the solution of problem **1367**, we then have $e_3 < 0 < e_2 < e_1$ and $k^2 > \frac{1}{2}$.

(2) $f(u) = \dfrac{M}{4\pi\omega}\dfrac{\theta'\left(\dfrac{u-\alpha}{2\omega}\right)}{\theta_1'\left(\dfrac{u-\alpha}{2\omega}\right)} + c$. For $\alpha = 0$ and $M = 2\pi$ the mapping $t = f(u) = \zeta(u) - \dfrac{\eta}{\omega}u = \dfrac{1}{2\omega}\dfrac{\theta_1'\left(\dfrac{u}{2\omega}\right)}{\theta_1\left(\dfrac{u}{2\omega}\right)}$ is shown in Fig. 132 $\left(\text{note that } e_3 < -\dfrac{\eta}{\omega} < e_2\right)$.

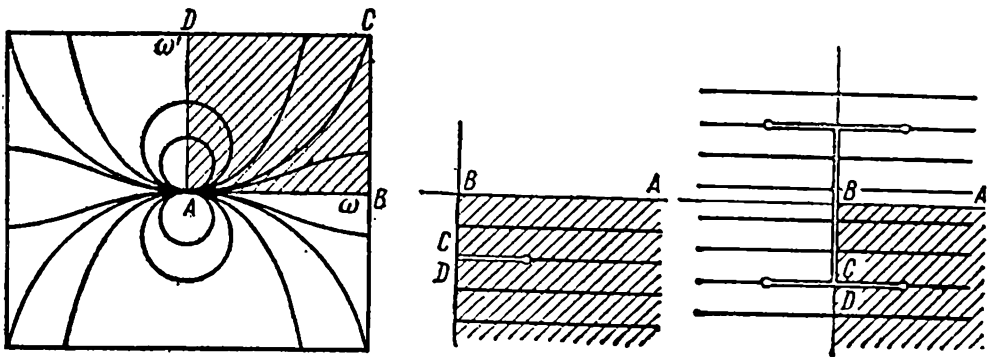

FIG. 132

1427. $f(u) = \dfrac{M}{2\pi}\zeta(u-\alpha) + \dfrac{N}{2\pi}\zeta(u-\beta) + Cu + c$. For the function $f(u)$ to be elliptic it must be of the form $f(u) = \dfrac{M}{2\pi}[\zeta(u-\alpha) - \zeta(u-\beta)] + c$. If $\operatorname{Im} u = \pm \operatorname{Im} \omega'$ is a stream line, then M must be real, and if they are equipotential lines, M must be purely imaginary. For α and β only the values 0, ω_k $(k = 1, 2, 3)$ are possible. For $\alpha = 0$, $\beta = \omega_k$, $M = 2\pi$

$$f(u) = \zeta(u) - \zeta(u-\omega_k) = \eta_k - \frac{1}{2}\frac{\mathfrak{p}'(u)}{\mathfrak{p}(u) - e_k} = \eta_k + \frac{\sigma_i(u)\,\sigma_j(u)}{\sigma(u)\,\sigma_k(u)}$$

(i, j, k are permutations of 1, 2, 3). The points $u \equiv \dfrac{\omega_k}{2} \pmod{\omega, \omega'}$ are

critical, that is, $f'(u) = 0$ there. For the basic mappings see Fig. 133. The rectangles given there are mapped onto the half-plane bounded by the horizontal straight line ($k = 1$), the half-plane bounded by the vertical

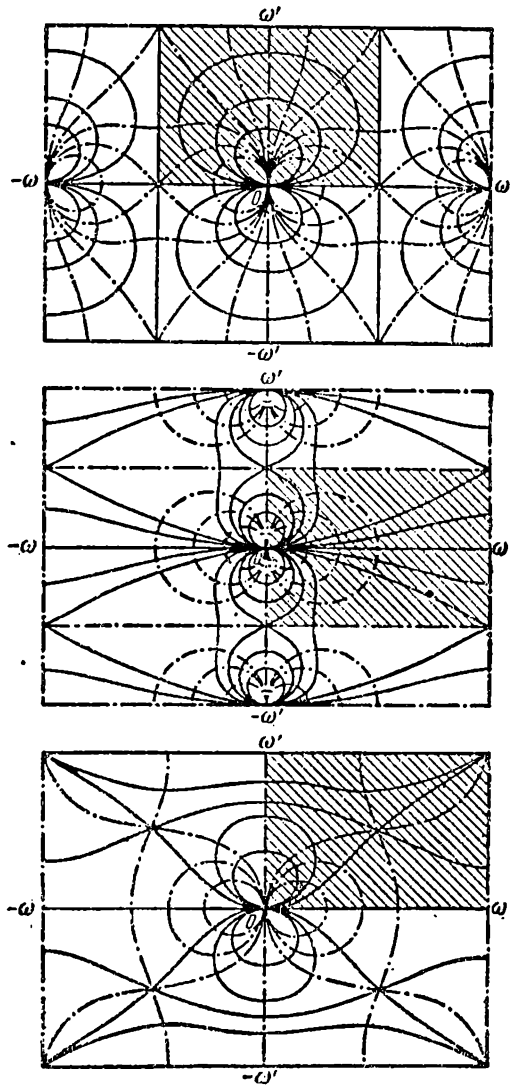

FIG. 133

straight line ($k = 2$), and onto a two-sheeted quadrant joined along the horizontal half-line corresponding to the dotted curve on the rectangle ($k = 3$). These mappings are continued by the symmetry principle.

1428. The periods of the flow are $4K$ and $2iK'$, doublets $2mK+(2n+1)iK'$ with moments $2\pi\left(-\frac{1}{k}\right)^m$, critical points $(2m+1)K+niK'$ (m,n integers). For the mapping see Fig. 134.

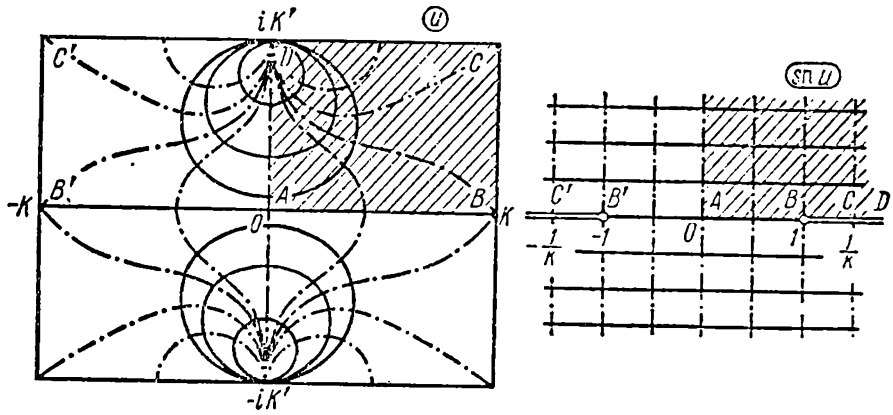

Fig. 134

1429. The periods of the flow are $4K$ and $2K+2iK'$, the doublets are the same as in problem **1428** with moments $2\pi\frac{(-1)^{n+m}}{ik}$, critical points $2mK+2niK'$ and $(2m+1)K+(2n+1)iK'$ (Fig. 135).

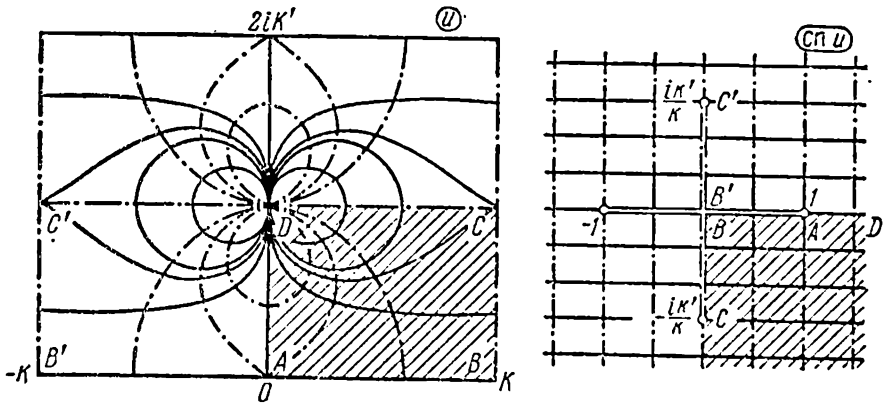

Fig. 135

1430. The periods of the flow are $2K$ and $4iK'$, the doublets are the same as in problem **1428** with the moments $2\pi(-1)^{n+m}i$, the critical points are $mK+2niK'$. The basic mappings are shown in Fig. 136.

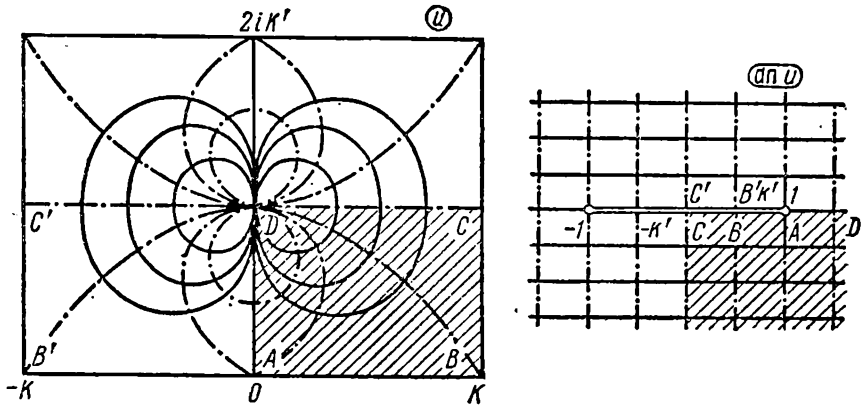

FIG. 136

1431. $f(u) = \frac{\Gamma+iQ}{2\pi i}\log\frac{\sigma(u-\alpha)}{\sigma(u-\beta)}+Cu+c$. In particular, for $\alpha = 0$, ω, $\omega+\omega'$, and $\beta = \omega'$ neglecting the additive constant and the factor $\frac{\Gamma+iQ}{2\pi i}$, varying C and transforming to the σ-functions, we obtain respectively, $\log\frac{\sigma(u)}{\sigma_3(u)}+Cu$, $\log\frac{\sigma_1(u)}{\sigma_3(u)}+Cu$, $\log\frac{\sigma_2(u)}{\sigma_3(u)}+Cu$. If $f(u+2\omega) = f(u)$ then

$$f(u) = \frac{\Gamma+iQ}{2\pi i}\log\frac{\theta_1\left(\frac{u-a}{2\omega}\right)}{\theta_1\left(\frac{u-\beta}{2\omega}\right)}+e.$$

1432. (1), (2), (3) doubly periodic flows with sources of strength 2π, and -2π at the zeros and poles of the functions sn u, cn u, dn u (Fig. 137).

1433. A doubly periodic flow with quadruplets at the zeros of $\mathfrak{p}(u)$ (Fig. 138).

1434. Periodic flows with the period 2ω (period of the velocity!) with sources of strength 2π at the zeros of $\theta_k(v)$ (Fig. 139, (1) for θ_4 and for θ_3 on displacing to the right by $\frac{1}{2}$; Fig. 139, (2) for θ_1 and for θ_2 on displacing to the right by $\frac{1}{2}$).

1435. (1) $f(z) = \frac{\Gamma}{2\pi i}\log z+c$; (2) $f(z) = \frac{\Gamma}{2\pi i}\log t(z)+c$, where $t(z)$ maps the domain D onto the circular ring with preservation of the directions of traversal of the boundary contours;

(3) $f(z) = \frac{\Gamma}{2\pi i}\log\frac{z-z_1}{z-z_2}+c$, where z_1, z_2 are mutually symmetrical with respect to each of the circles (that is, they are the points of intersection

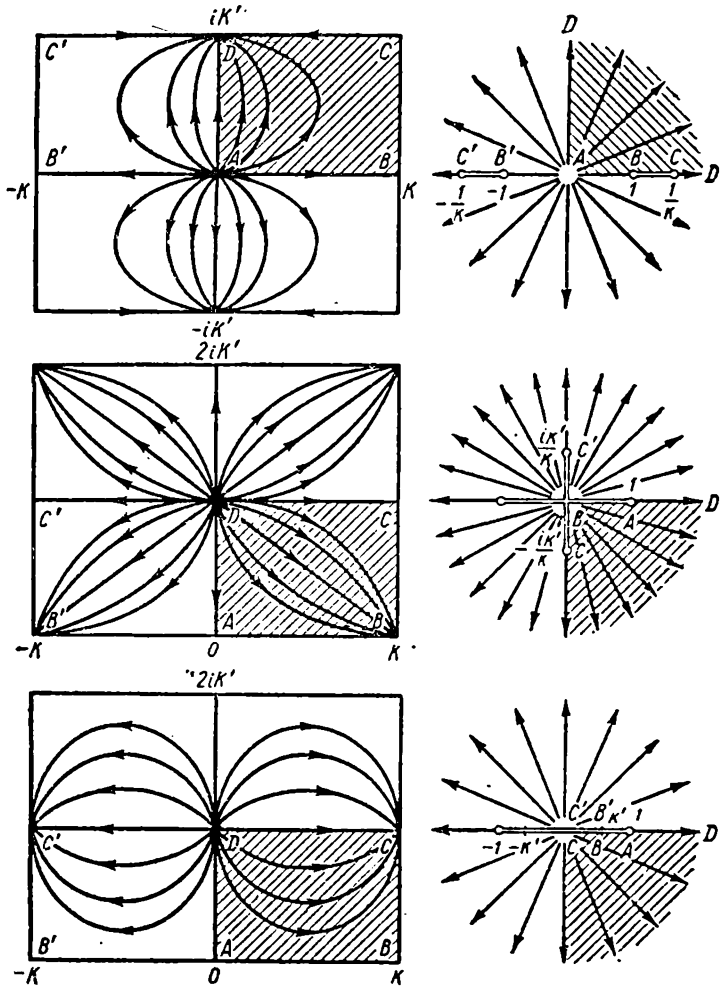

Fig. 137

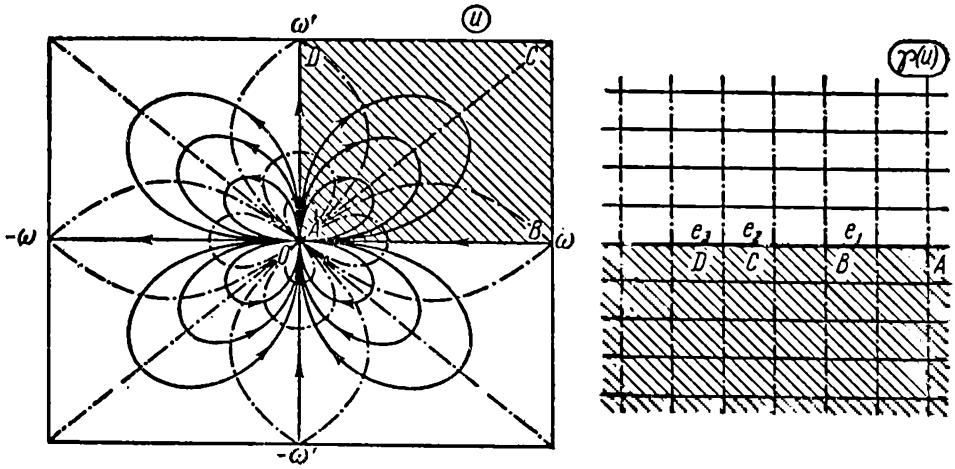

Fig. 138

of circles orthogonal to the two given circles with the straight line joining their centres), where the point z_1 lies inside the circle with circulation Γ;

(4) $f(z) = \frac{\Gamma}{2\pi i} \log t(z) + c$, where the function $t(z)$ maps the domain D onto the ring with preservation of the direction of traversal of the contour with circulation Γ.

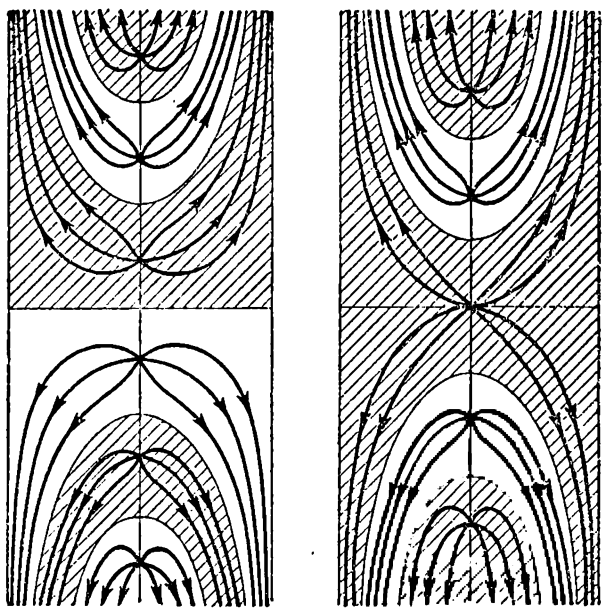

FIG. 139

1436. $f(z) = \Phi\left(\frac{\omega}{\pi i} \log z\right)$, where

$$\Phi(u) = -\frac{1}{4\pi^2 a}\left[p\frac{\theta_1'\left(\frac{u-\alpha}{2\omega}\right)}{\theta_1\left(\frac{u-\alpha}{2\omega}\right)} - \bar{p}\frac{\theta_1'\left(\frac{u+\alpha}{2\omega}\right)}{\theta_1\left(\frac{u+\alpha}{2\omega}\right)}\right] + c, \quad a = \frac{\omega}{\pi i}\log a.$$

The function $f(z)$ maps R onto the exterior of two parallel segments, at a distance $\frac{|\mathrm{Re}\, ip|}{2\pi a}$ from one another (Fig. 140). The ends of the segments are determined from the condition $\Phi'(u) = 0$.

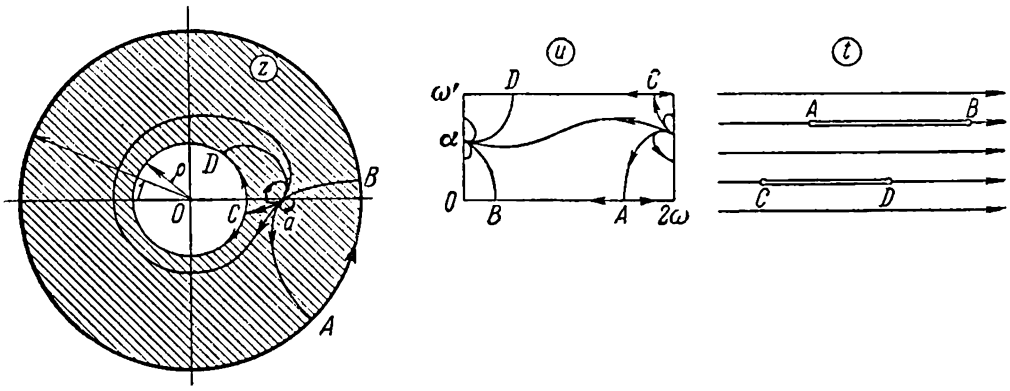

FIG. 140

1437. $f(z) = \Phi[u(z)]$, where

$$\Phi(u) = \frac{V}{2\omega}\left[ke^{i\alpha}\frac{\theta_1'\left(\frac{u-a}{2\omega}\right)}{\theta_1\left(\frac{u-a}{2\omega}\right)} + \bar{k}e^{-i\alpha}\frac{\theta_1'\left(\frac{u-\bar{a}}{2\omega}\right)}{\theta_1\left(\frac{u-\bar{a}}{2\omega}\right)}\right] + c,$$

and $u(z) = a + \dfrac{k}{z} + \ldots$ is the function which maps the domain D onto the rectangle.

1438. $f(z) = \Phi\left(\dfrac{\omega}{\pi i}\log z\right)$, where

$$\Phi(u) = A\mathfrak{p}(u) + \frac{B}{2\omega}\frac{\theta_1'\left(\frac{u}{2\omega}\right)}{\theta_1\left(\frac{u}{2\omega}\right)} + c,$$

$A = -\dfrac{c_{-2}\omega^2}{\pi^2}$, $B = \dfrac{i\omega}{\pi}(c_{-2} - c_{-1})$. The problem is possible if c_{-2} is a real number and the difference $c_{-2} - c_{-1}$ is purely imaginary. If $A \neq 0$, then the function $f(z)$ maps R onto the exterior of a horizontal ray and a segment parallel to it and distant $\dfrac{|c_{-2} - c_{-1}|}{2}$ from it. The ends of the segment and the beginning of the ray are determined from the condition $\Phi'(u) = 0$ (Fig. 141, (1)), (the case $B = 0$ is shown in Fig. 141, (2)). If, however, $A = 0$, that is, there is only a doublet then R is mapped onto the half-plane bounded by the horizontal straight line and having a cut along a horizontal segment at a distance $\dfrac{|c_{-1}|}{2}$ from the straight line (Fig. 141, (3)).

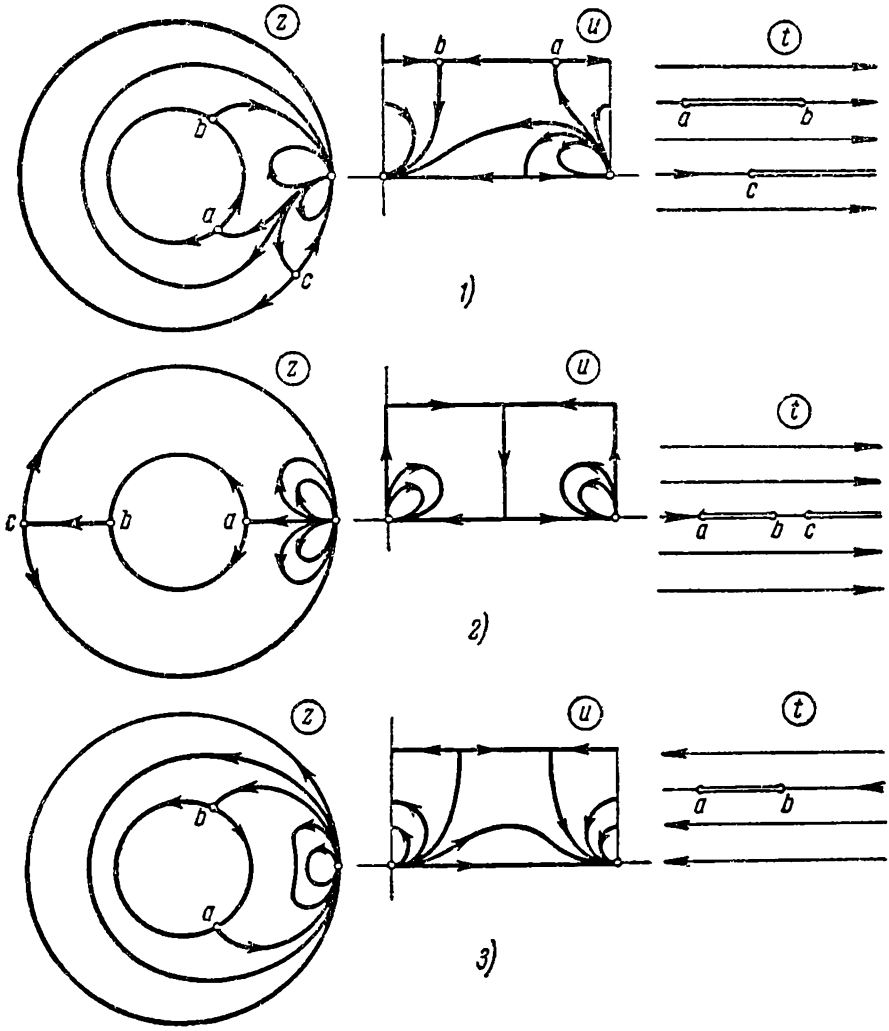

Fig. 141

1439. (1) A solution is possible if $\Gamma_1-\Gamma_2=\Gamma$; with this condition $f(z)=\Phi\left(\frac{\omega}{\pi i}\log z\right)$, where

$$\Phi(u)=\frac{\Gamma}{2\pi i}\log\frac{\theta_1\left(\frac{u-\alpha}{2\omega}\right)}{\theta_1\left(\frac{u+\alpha}{2\omega}\right)}+\frac{\Gamma_2}{2\omega}u+c,\quad a=\frac{\omega}{\pi i}\log a$$

(it is necessary to bear in mind that the increment of $\log\frac{\left[\theta_1\left(\frac{u-\alpha}{2\omega}\right)\right]}{\left[\theta_1\left(\frac{u+\alpha}{2\omega}\right)\right]}$ on varying u from 0 to 2ω equals $2\pi i$, and on varying u from $2\omega+i\omega'$ to $i\omega'$ it equals 0). The stagnation points of the flow are determined from the equation

$$\mathfrak{p}(u)=\mathfrak{p}(\alpha)+\frac{1}{2}\frac{\mathfrak{p}'(\alpha)}{\left[\zeta(\alpha)-\frac{\eta}{\omega}\alpha\right]+\lambda\left[\zeta(\omega')-\frac{\eta}{\omega}\omega'\right]},$$

where $\lambda=\Gamma_2/\Gamma$, and are situated on the sides of the rectangle with vertices $(0,\ \omega,\ \omega+\omega',\ \omega')$ and the rectangles symmetrical to it. In the case $\Gamma_2=0$ the function $\mathfrak{Z}(z)$ maps R onto a circle with a cut along an arc of the circumference (Fig. 142; $\Gamma>0$). In the case $\Gamma_2=-\Gamma_1=-\frac{\Gamma}{2}$ the function

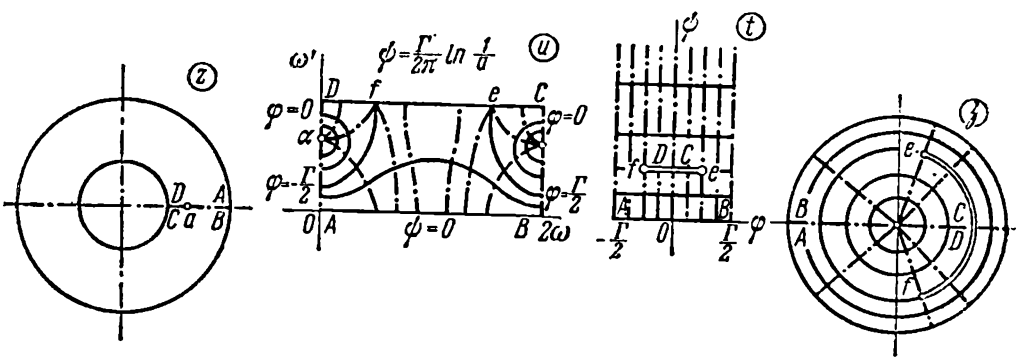

FIG. 142

$\mathfrak{Z}(z)$ maps R onto the two-sheeted domain formed by joining the exteriors of the circles $|\mathfrak{Z}|>1$ and $|\mathfrak{Z}|>\frac{\varrho}{a^2}$ along the segments from $-\infty$ to $\mathfrak{Z}_0=-e^{\frac{4\pi\psi_0}{\Gamma}}$ where ψ_0 is the value of ψ at the stagnation point. The function $s(\mathfrak{Z})$ maps this two-sheeted domain onto the exterior of two lemniscates, and

$f(z) = -\frac{\Gamma}{4\pi i}\log[(s(z)-s_0)(s(z)+s_0)]$ (Fig. 143; $\Gamma > 0$, $a^2 < \varrho$; it is necessary to join the half-strips in the t-plane along the common cut). In the general case $\psi = 0$ on the lower base of the rectangle in the u-plane and ϕ varies

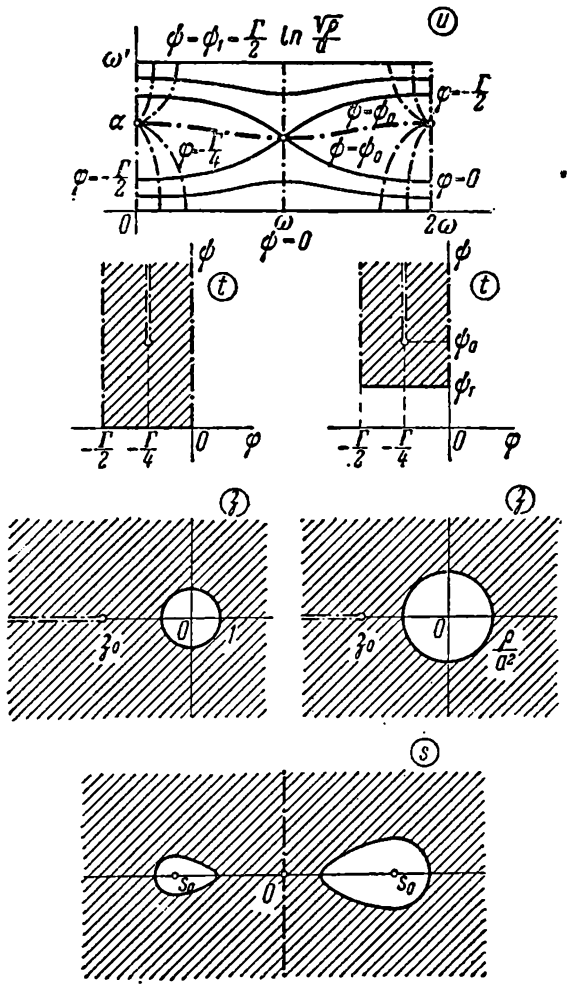

FIG. 143

from $-\frac{\Gamma}{2}$ to $\frac{\Gamma}{2}+\Gamma_2$, and on the upper base $\psi = \frac{\Gamma}{2\pi}\log\frac{1}{a}+\frac{\Gamma_2}{2\pi}\log\frac{1}{\varrho}$ and ϕ is varied from 0 to Γ_2 ($\Gamma > 0$).

(2) $f(z) = \Phi[u(z)]$, where

$$\Phi(u) = \frac{V}{2\omega}\left[ke^{i\alpha}\frac{\theta_1'\left(\frac{u-a}{2\omega}\right)}{\theta_1\left(\frac{u-a}{2\omega}\right)} + \bar{k}e^{-i\alpha}\frac{\theta_1'\left(\frac{u-\bar{a}}{2\omega}\right)}{\theta_1\left(\frac{u-\bar{a}}{2\omega}\right)}\right]$$

$$+ \frac{\Gamma_1-\Gamma_2}{2\pi i}\log\frac{\theta_1\left(\frac{u-a}{2\omega}\right)}{\theta_1\left(\frac{u-\bar{a}}{2\omega}\right)} + \frac{\Gamma_2}{2\omega}u + C,$$

and $u(z) = a + \dfrac{k}{z} + \ldots$ maps the domain D onto a circular ring.

1450. The trajectories of the vortex are closed curves within the rectangle with centre of symmetry at the point $a+bi$. For a vortex close to the centre of symmetry these trajectories are nearly ellipses. If the vortex is at the point $a+bi$ itself these trajectories are stationary.

Solution. Let $z_0 = x_0+iy_0$ be the position of the vortex in the rectangle considered. Continuing the flow by the symmetry principle we obtain a doubly periodic flow with periods $2\omega = 4a$, $2\omega' = 4bi$, having in the rectangle of periods two vortices of intensity Γ at the points $\pm z_0$ and two vortices of intensity $-\Gamma$ at the points $\pm \bar{z}_0$. Hence

$$\omega(z) = \frac{\Gamma}{2\pi i}\log\frac{\theta_1\left(\frac{z-z_0}{4a}\right)\theta_1\left(\frac{z+z_0}{4a}\right)}{\theta_1\left(\frac{z-\bar{z}_0}{4a}\right)\theta_1\left(\frac{z+\bar{z}_0}{4a}\right)},$$

consequently,

$$w'(z) = \frac{\Gamma}{2\pi i}\left\{\left[\zeta(z-z_0) - \frac{\eta}{2a}(z-z_0)\right] + \left[\zeta(z+z_0) - \frac{\eta}{2a}(z+z_0)\right]\right.$$

$$\left. - \left[\zeta(z-\bar{z}_0) - \frac{\eta}{2a}(z-\bar{z}_0)\right[- \left[\zeta(z+\bar{z}_0) - \frac{\eta}{2a}(z+\bar{z}_0)\right]\right\}.$$

Subtracting from this $\dfrac{\Gamma}{2\pi i}\dfrac{1}{z-z_0}$ and passing to the limit as $z \to z_0$, we obtain:

$$\frac{dx_0}{dt} - i\frac{dy_0}{dt} = \frac{\Gamma}{2\pi i}[\zeta(2x_0+2iy_0) - \zeta(2iy_0) - \zeta(2x_0)] = \frac{\Gamma}{4\pi i}\frac{\mathfrak{p}'(2x_0)-\mathfrak{p}'(2iy_0)}{\mathfrak{p}(2x_0)-\mathfrak{p}(2iy_0)},$$

whence

$$\left.\begin{aligned}\frac{dx_0}{dt} &= i\frac{\Gamma}{4\pi}\frac{\mathfrak{p}'(2iy_0)}{\mathfrak{p}(2x_0)-\mathfrak{p}(2iy_0)},\\ \frac{dy_0}{dt} &= \frac{\Gamma}{4\pi}\frac{\mathfrak{p}'(2x_0)}{\mathfrak{p}(2x_0)-\mathfrak{p}(2iy_0)}\end{aligned}\right\} \qquad (*)$$

(since $\mathfrak{p}(2iy_0)$ is a real number, $\mathfrak{p}'(2iy_0)$ is a purely imaginary number). (From (*) it follows that $\mathfrak{p}'(2x_0)dx_0 = i\mathfrak{p}'(2iy_0)\,dy_0$. But for the analytic function $\mathfrak{p}(z)$ the derivative $\mathfrak{p}'(z) = \dfrac{\partial \mathfrak{p}}{\partial x} = \dfrac{1}{i}\dfrac{\partial \mathfrak{p}}{\partial y}$ and it follows from the preceding that $d\mathfrak{p}(2x_0) = d\mathfrak{p}(2iy_0)$, consequently,

$$\mathfrak{p}(2x_0) - \mathfrak{p}(2iy_0) = C. \qquad (**)$$

Such are the trajectories of the vortex within the rectangle. In order to investigate them, let us note, that when x_0 increases from 0 to a, the value of $\mathfrak{p}(2x_0)$ decreases from $+\infty$ to $e_1 = \mathfrak{p}(2a)$, then increases to $+\infty$, and when y_0 increases from 0 to b, the value of $\mathfrak{p}(2iy_0)$ increases from $-\infty$ to $e_3 = \mathfrak{p}(2bi)$, then decreases to $-\infty$. We conclude from this that for $C > e_1 - e_3$ the trajectories (**) represent closed curves symmetrical about the point $a+bi$; for $C = e_1 - e_3$ the curve shrinks to a point — the stationary case. In particular, if $x_0 = a+\xi$, $y_0 = b+\eta$ and ξ, η are small, then $\mathfrak{p}(2a+2\xi) \approx e_2 + \dfrac{1}{2!}(2\xi)^2\mathfrak{p}''(2a)$, as $\mathfrak{p}'(2a) = 0$; $\mathfrak{p}(2bi+2\xi i) \approx e_3 + \dfrac{1}{2!}(2\eta i)^2\mathfrak{p}''(2bi)$, as $\mathfrak{p}'(2bi) = 0$ and the equation (**) assumes the form $\mathfrak{p}''(2a)\xi^2 + \mathfrak{p}''(2bi)\eta^2 \approx \dfrac{1}{2}[C - (e_1 - e_3)]$ that is the trajectories of the vortex are close to ellipses.

1451. $u = \alpha x - \beta y$, $v = \beta x + \alpha y$, $\mathbf{E} = -i\bar{c}$; a dipole $(\infty;\ -i\bar{c})$.

1452. $u = 2q\phi$, $v = 2q \log \dfrac{1}{r}$; $\mathbf{E} = \dfrac{2q}{r} e^{i\phi}$; point charges $(a;\ 2q)$ and $(\infty;\ -2q)$.

1453. $u = 2q \arg \dfrac{z-b}{z-a}$, $v = 2q \log \left|\dfrac{z-a}{z-b}\right|$; $\mathbf{E} = \dfrac{2q(\bar{b}-\bar{a})}{(\bar{z}-\bar{a})(\bar{z}-\bar{b})}$;

point charges $(b;\ 2q)$ and $(a;\ -2q)$.

1454. $u = -2q \arg (z^2-a^2)$, $v = 2q \log |z^2-a^2|$; $\mathbf{E} = -4q\bar{z}/(\bar{z}^2-\bar{a}^2)$; point charges $(a;-2q)$, $(-a;\ -2q)$ and $(\infty;\ 4q)$ (see Fig. 122).

1455. $u = \dfrac{|p|}{r}\sin(\phi-\alpha)$, $v = \dfrac{|p|}{r}\cos(\phi-\alpha)$; $\mathbf{E} = \dfrac{2|p|}{r^2} e^{i(2\phi-\alpha)}$;

a dipole $(0;\ p)$ (Fig. 144).

1456. $u = \left(r \pm \dfrac{R^2}{r}\right)\cos\phi$, $v = \left(r \pm \dfrac{R^2}{r}\right)\sin\phi$; $\mathbf{E} = -i\left(1 \mp \dfrac{R^2}{r^2} e^{2i\phi}\right)$;

dipoles $(0;\ \mp iR^2)$ and $(\infty;\ -i)$ (see Fig. 119, 120).

1457. $u = -py + 2q\phi$, $v = px + 2q \log \dfrac{1}{r}$; $\mathbf{E} = -p + \dfrac{2q}{r} e^{i\phi}$;

point charges $(0;\ 2q)$ and $(\infty;-2q)$; a dipole $(\infty;\ p)$ (see Fig. 127).

1458. $u = -py + \sum_{k=1}^{n} 2q_k\phi_k$, $v = px + \sum_{k=1}^{n} 2q_k \log \dfrac{1}{r_k}$;

$$\mathbf{E} = -p + \sum_{k=1}^{n} \frac{2q_k}{r_k} e^{i\phi_k},$$

where $z - a_k = r_k e^{i\phi_k}$; point sources $(a_k;\ 2q_k)$; a dipole $(\infty;\ p)$ (Fig. 145).

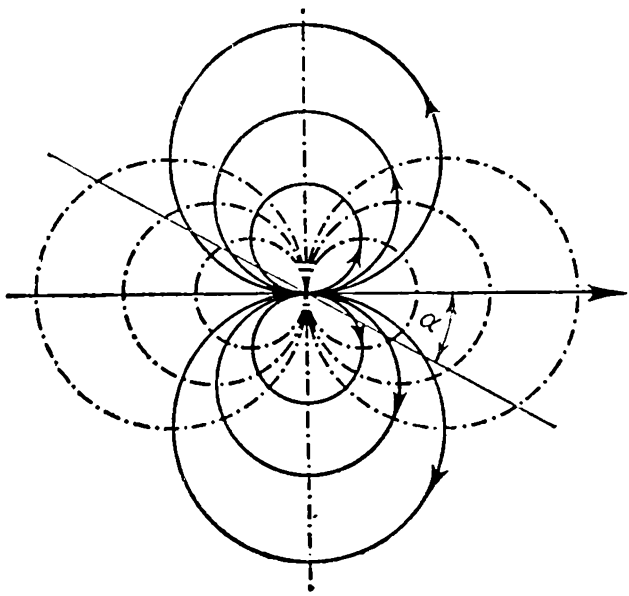

FIG. 144

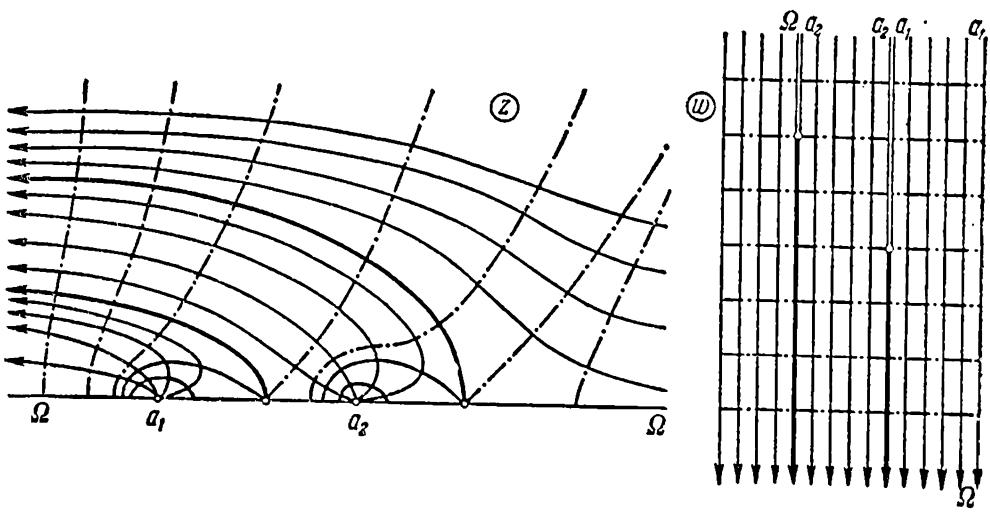

FIG. 145

1459. (1) The value of the point charge is preserved; the law of variation of the moment of the dipole is the same as in problem **1393**;

(2) The sign of the charge is reversed; the law of variation of the moment of the dipole is the same as in problem **1395** on continuation across the equipotential line.

1460. $v = 2q\,g(z,a)$.

1461. $w = 2qi \log \dfrac{z-\bar{z}_0}{z-z_0} + c$.

1462. (1) and (2) $w = 2qi \log \dfrac{R^2-\bar{z}_0 z}{R(z-z_0)} + c$.

1463. $w = 2qi \log \dfrac{1}{f(z)} + \text{const}$, where $f(z) = \dfrac{z-\sqrt{(z^2-c^2)}}{a-b}$, $c^2 = a^2-b^2$.

1464. $w = 2qi \log \dfrac{1}{f(z)} + c$, where $f(z) = [z-\sqrt{(z^2-R^2)}]/R$.

1465. $w = 2qi \log \dfrac{1}{f(z)}$, where $t = f(z)$ is determined from the equation $z = \dfrac{2d}{B\left(\frac{1}{2}, \frac{3}{4}\right)} \displaystyle\int_t^1 \frac{\sqrt{(1-t^4)}}{t^2}\,dt + \frac{d}{2}$ (see problem **1338** for $n = 4$ and **1374** for $k = 1/\sqrt{2}$).

1466. $w = 2qi \log \dfrac{1}{f(z)} + c$, where $f(z) = \dfrac{1-\operatorname{cn}\left(\frac{K}{a}z, k\right)}{\operatorname{sn}\left(\frac{K}{a}z, k\right)}$ and k is determined from the equation $K'/K = b/a$ (see problem **1371**).

1467. $w = 2qi \log \dfrac{1}{f(z)} + c$, where $f(z) = \dfrac{\theta_1\left(\frac{z-z_0}{4a}\right)\theta_1\left(\frac{z-z_2}{4a}\right)}{\theta_1\left(\frac{z-z_1}{4a}\right)\theta_1\left(\frac{z-z_3}{4a}\right)}$

and $\omega = 2a$, $\omega' = 2ib$, $z_1 = (4a-x_0)+iy_0$, $z_2 = (4a-x_0)+i(4b-y_0)$, $z_3 = x_0+i(4b-y_0)$ (see problem **1450**).

1468. $w = \dfrac{pi}{z-a} + \dfrac{p^*i}{z-a^*} + c \quad \left(a \neq 0, a^* = \dfrac{R^2}{\bar{a}}, p^* = \dfrac{R^2\bar{p}}{\bar{a}^2}\right)$,

$w = \dfrac{pi}{z} - \dfrac{\bar{p}i}{R^2}z + c$ $(a = 0)$, c is a real number. Compare with problem **1368**, (2).

1469. $w = \dfrac{pi}{z-a} + \dfrac{p^*i}{z-a^*} + c \quad \left(a \neq \infty, a^* = \dfrac{R^2}{\bar{a}}, p^* = \dfrac{R^2}{\bar{a}^2}\bar{p}\right)$,

$w = \dfrac{pi}{z} - \dfrac{\bar{p}i}{R^2}z + c$ $(a = \infty)$. See problem **1400**, (2).

1470. $w = \varrho(z\cos\alpha + i\sin\alpha\sqrt{(z^2-R^2)}) + \text{const}$.

1471. $w = \dfrac{\varrho}{a-b}\left[(az-b\sqrt{(z^2-c^2)})\cos\alpha - i(bz-a\sqrt{(z^2-c^2)})\sin\alpha\right] + \text{const}$

where $c^2 = a^2-b^2$.

1472. $w = \dfrac{2K\varrho}{a\,\mathrm{sn}\,u}(\cos\alpha + i\sin\alpha\,\mathrm{cn}\,u)$, where $u = \dfrac{K}{a}z$ and $\dfrac{K'}{K} = \dfrac{b}{a}$ (see problem **1466**).

1473. (1) If $pi = \varrho e^{i\alpha}$, then

$$f(z) = \varrho t'(a)\left\{\left[\frac{1}{t(z)} + t(z)\right]\cos\alpha + i\left[\frac{1}{t(z)} - t(z)\right]\sin\alpha\right\} + c;$$

(2) $f(z) = pit(z) - \dfrac{R^2\bar{p}i}{t(z)} + c = \varrho\left\{\left[t(z) + \dfrac{R^2}{t(z)}\right]\cos\alpha + i\left[t(z) - \dfrac{R^2}{t(z)}\right]\sin\alpha\right\} + c,$

where $pi = \varrho e^{i\alpha}$. The functions in the square brackets effect the normalised conformal mapping of D onto the exterior of the horizontal and vertical segments respectively†.

1474. $w = \displaystyle\sum_{k=1}^{n} 2q_k i \log\frac{1}{f(z, a_k)} + if'(a, a)\left[\frac{p}{f(z, a)} - \bar{p}f(z, a)\right] + c,$

where $f(z, a_k)$ and $f(z, a)$ conformally map D onto the unit disk with the normalisation $f(a_k, a_k) = f(a, a) = 0$, $f'(a, a) > 0$ and c is a real number.

1476. $v(z, a) = \begin{cases} -\log\dfrac{1}{|z-\bar{a}|}, & \text{if } \operatorname{Im} z \geqslant 0, \\ -\log\dfrac{1}{|z-a|}, & \text{if } \operatorname{Im} z \leqslant 0; \end{cases}$

$$\varrho(x, a) = -\frac{1}{\pi}\frac{\beta}{(x-\alpha)^2+\beta^2} \quad (a = \alpha+i\beta).$$

1477. (1) Inside the circle

$$v(z, a) = \log\frac{|R^2-\bar{a}z|}{R} = \begin{cases} -\left[\log\dfrac{1}{\left|z-\dfrac{R^2}{\bar{a}}\right|} + \log\dfrac{|a|}{R}\right] & \text{if } a \neq 0, \\ \log R, & \text{if } a = 0, \end{cases}$$

Outside the circle $v(z, a) = -\log\dfrac{1}{|z-a|}$. The density is

$$\varrho(Re^{i\theta}, a) = -\frac{1}{2\pi R}\frac{R^2-|a|^2}{R^2-2R|a|\cos(\theta-\alpha)+|a|^2} \quad (a = |a|e^{i\alpha}).$$

In particular for $a = 0$ it has the constant value $-\dfrac{1}{2\pi R}$ and is identical

† See the appendix by M. Schiffer to the book: R. COURANT, *Dirichlet's Principle, Conformal Mappings and Minimal Surfaces*, § 1, sec. 2, especially pages 242–3, Interscience, New York, 1950.

with the potential of a layer having the constant value $\log R$ inside the circle and the value $\log |z|$ outside the circle.

$$(2)\quad v(z, a) = \begin{cases} \log \dfrac{|R^2 - \bar{a}z|}{R}, & a \neq \infty,\ |z| \geqslant R, \\ -\log \dfrac{1}{|z-a|}, & a \neq \infty,\ |z| \leqslant R; \end{cases}$$

$$v(z, \infty) = \begin{cases} \log |z|, \text{ if } |z| \geqslant R, \\ \log R, \text{ if } |z| \leqslant R; \end{cases}$$

$$\varrho(Re^{i\theta}, a) = -\frac{1}{2\pi R} \frac{|a|^2 - R^2}{R^2 - 2R|a| \cos(\theta - \alpha) + |a|^2} \quad (a = |a|e^{i\alpha} \text{ and } a \neq \infty).$$

If $a = \infty$ the same potential is induced as in the preceding case for $a = 0$.

1478. $v(z, \infty) = \log \dfrac{|z + \sqrt{(z^2 - R^2)}|}{2}$, $\varrho(x, \infty) = -\dfrac{1}{2\pi\sqrt{(R^2 - x^2)}}$ $(|x| < R)$.

1479. $v(z, \infty) = \log \dfrac{|z + \sqrt{(z^2 - c^2)}|}{2}$, outside the ellipse, $v(z, \infty) = -\log 2(\alpha - \beta)$ inside the ellipse. The density is $\varrho(z, \infty) = -\dfrac{1}{2\pi\sqrt{(|\zeta^2 - c^2|)}}$ (ζ is on the ellipse, $c^2 = \alpha^2 - \beta^2$).

1480. $\varrho(\zeta) = \dfrac{1}{2\pi} \dfrac{\partial g(\zeta, \infty)}{\partial n}$.

1481. $\varrho(\zeta) = \dfrac{1}{2\pi R}$.

1482. $\varrho(x) = \dfrac{1}{2\pi\sqrt{(R^2 - x^2)}}$ $(|x| < R)$.

1483. $\varrho(\zeta) = \dfrac{1}{2\pi\sqrt{(|\zeta^2 - c^2|)}}$ $(c^2 = a^2 - b^2)$.

1484. R.

1485. $1/2R$.

1486. $1/2(a - b)$.

1487. a.

1490. If $\omega(z; \Delta)$ is the harmonic measure of the interval Δ of the real axis at the point z with respect to the upper half-plane: $\omega(z; \Delta) = \dfrac{1}{\pi} \int\limits_{\Delta} \dfrac{\partial}{\partial n} \log \dfrac{1}{|t - z|}\, dt$ (see problems **1091** and **1489**), then (omitting a real additive constant): (1) $w = \dfrac{\phi}{\pi} \log(z - a)$, $v = \phi\omega(z; \Delta)$, $\Delta = (-\infty, a)$; (2) $w = \dfrac{\phi}{\pi} \log \dfrac{z - b}{z - a}$, $v = \phi\omega(z; \Delta)$, $\Delta = (a, b)$; (3) $w = \dfrac{1}{\pi}\left[\phi_1 \log(z - a_1) + \phi_2 \log \dfrac{z - a_2}{z - a_1} + \ldots + \phi_n \log \dfrac{z - a_n}{z - a_{n-1}}\right]$, $v = \sum\limits_{k=1}^{n} \phi_k \omega(z; \Delta_k)$, $\Delta_k = (a_{k-1}, a_k)$, $a_{-1} = -\infty$; (4) w and v are obtained from the expressions indicated in the answer to part (3) by replacing ϕ_k by $\phi_k - \phi_0$, by the addition of $i\phi_0$ to w and of ϕ_0 to v. The potential can also be represented in the form $\sum\limits_{k=0}^{n} \phi_k \omega(z; \Delta_k)$, $\Delta_0 = (a_n, \infty)$.

1491. (1) $w(z) = \dfrac{id}{\log \mu} \log z + \text{const}$, $v(z) = \dfrac{d}{\log \mu} \log r + \text{const}$, where $\mu = \dfrac{r_2}{r_1}$;

(2) $w(z) = \dfrac{id}{\log \mu} \log t(z) + \text{const}$, $v(z) = \dfrac{d}{\log \mu} \log |t(z)| + \text{const}$, where μ is the modulus of the domain D (see page 30) and $t(z)$ conformally maps D onto the circular ring.

1493. $w(z)$ is determined by the formula indicated in the text of problem **1492**, where: (1) $t(z) = \sqrt{\mu}\, \dfrac{z - \sqrt{(a^2 - R^2)}}{z + \sqrt{(a^2 - R^2)}}$, $\mu = \dfrac{a}{R} + \sqrt{\left[\left(\dfrac{a}{R}\right)^2 - 1\right]}$;

(2) and (3) $t(z) = \sqrt{(\mu)}\, \dfrac{z - x_1}{z - x_2}$, $\mu = \left(\dfrac{R - x_1}{R - x_2}\right)^2$, x_1, x_2 are determined from the equation $x^2 + \dfrac{r^2 - R^2 - a^2}{a} x + R^2 = 0$ $(x_1 < x_2)$;

(4) $t(z) = \dfrac{1}{c}\,(z + \sqrt{(z^2 - c^2)})$, $\mu = \dfrac{a+b}{c}$, $c = \sqrt{(a^2 - b^2)}$;

(5) $t = e^{\pi \frac{K+u}{K'}}$, $z = \operatorname{sn}(u, k)$, $\mu = e^{2\pi \frac{K}{K'}}$ $(|\operatorname{Re} u| < K,\ |\operatorname{Im} u| \leqslant K')$ (see problem **1370**, (1));

(6) $t = e^{\pi \frac{K+u}{K'}}$, $z = \dfrac{2Kh}{\pi}\left[Z(u) + \dfrac{\operatorname{dn} u \operatorname{cn} u}{\operatorname{sn} u}\right]$, $\mu = e^{2\pi \frac{K}{K'}}$

where k is determined from the equations: $KZ(\mu) = \dfrac{\pi h}{2a}$, $\operatorname{dn}^2 \beta = \dfrac{E}{K}$ (see problem **1370**, (15)).

1495. (1) $\omega_1(z) = 1 - \dfrac{\log |z|}{\log \mu}$, $\omega_2(z) = \dfrac{\log |z|}{\log \mu}$;

$$\varrho_1(\zeta) = -\varrho_2(\zeta) = \begin{cases} \dfrac{1}{2\pi\mu \log \mu} & \text{on } |z| = 1, \\ -\dfrac{1}{2\pi\mu \log \mu} & \text{on } |z| = \mu; \end{cases} \qquad p_{11} = p_{22} = -p_{12} = -p_{21} = \frac{1}{\log \mu};$$

(2) $\omega_1(z) = 1 - \omega_2(z)$, $\omega_2(z) = \dfrac{\log |t(z)|}{\log \mu}$;

$$\varrho_1(\zeta) = -\varrho_2(\zeta) = \begin{cases} \left|\dfrac{t'(\zeta)}{t(\zeta)}\right| \dfrac{1}{\log \mu} & \text{on } \Gamma_1, \\ -\left|\dfrac{t'(\zeta)}{t(\zeta)}\right| \dfrac{1}{\log \mu} & \text{on } \Gamma_2; \end{cases} \qquad p_{11} = p_{22} = -p_{12} = -p_{21} = \frac{1}{\log \mu},$$

where $t(z)$ conformally maps D onto the ring $1 < |t| < \mu$, the contour Γ_1 corresponding to the circle $|t| = 1$.

1497. $v(z) = \operatorname{Im} f(z) - \sum_{k=1}^{n} \alpha_k \omega_k(z)$, where $f(z)$ conformally maps D onto the plane with horizontal cuts, and

$$f(z) = \begin{cases} \dfrac{pi}{z-a} + \ldots, & \text{if } a \neq \infty, \\ piz + \ldots, & \text{if } a = \infty. \end{cases}$$

1498. $v(z) = 2qg(z, a) + \sum_{k=1}^{n} \alpha_k \omega_k(z)$, $\omega_k(z) = -\dfrac{1}{2k} \int_{\Gamma_k} \dfrac{\partial g(\zeta, z)}{\partial n} ds$.

1499. If $f(a) = \infty$, then the field is formed by the dipole $(a; p)$ where p is determined from the expansion of $f(z)$ close to the point a;

$$f(z) = \begin{cases} \dfrac{pi}{z-a} + \ldots, & \text{if } a \neq \infty, \\ piz + \ldots, & \text{if } a = \infty. \end{cases}$$

1500. (1) If $f(a) = 0$, $f(b) = \infty$, then the field is formed by the point charges $(a; 2q)$, $(b; -2q)$ the flux of the vector intensity through each boundary contour being equal to zero;

(2) If $f(a) = 0$, then the field is formed by the point charge $(a, 2q)$, the flux of the vector intensity through a boundary contour, corresponding to a circle, in the direction of the outward normal to the domain D being equal to $4\pi q$, and through every other contour being equal to zero;

(3) The field is regular everywhere. The flux of the vector intensity through boundary contours, transformed into circles, in the direction of the outward normal to D is equal to $\pm 4\pi q$ (+ for the contour, transformed into the outer circle), and through every other contour it is equal to zero.

1501. See problems **1437** and **1439**.

1502. (1) $v(z) = \sum_{k=1}^{n-1} \alpha_k \omega_k(z) + c$, where the α_k are uniquely determined from the system $\sum_{k=1}^{n-1} p_{ik}\alpha_k = 2q_i$ $(i = 1, 2, \ldots, n-1)$ (see problem **1104**) and c is an arbitrary real number. The problem is equivalent to the construction of the flow in D, streamlining the boundary contours Γ_k with circulations $4\pi q_k$ $(k = 1, 2, \ldots, n)$, if $\infty \in D$, and by the circulations $4\pi q_k$ $(k = 1, 2, \ldots, n-1)$, $-4\pi q_n$, if $\infty \overline{\in} D$ (Γ_n is the external contour).

(2) $v(z) = v_0(z) - 2qg(z, a)$, where $v_0(z)$ is determined as in part (1), form the charges of the layer $2q_k + 2q_k'$, where

$$q_k' = -\frac{q}{2\pi} \int_{\Gamma_k} \frac{\partial g(\zeta, a)}{\partial n} ds.$$

1503. $v(z) = 2q_1 \log \dfrac{r_2}{|z|} + c$, if $q = 0$, and

$$v(z) = 2(q_1 - \lambda q) \log \frac{r_2}{|z|} - 2qg(z, a) + c,$$

where $\lambda = \dfrac{\log a}{\log \varrho}$, $g(z, a) = |z|^{\lambda} \log \left| \dfrac{\theta_1\left(\dfrac{\log z + \log a}{2\pi i}\right)}{\theta_1\left(\dfrac{\log z - \log a}{2\pi i}\right)} \right|$

for $\tau = (\log \varrho)/\pi i$, where $\varrho < a < 1$, if $q \neq 0$ (in the notation of problem **1439** the Green's function $g(z, a) = \operatorname{Im} \Phi\left(\dfrac{\omega}{\pi i} \log z\right)$ for $\Gamma = 2\pi$ and $\Gamma_2 = -2\pi$; the latter from the condition $\psi = 0$ on the boundary of the ring).

1504. The source $(a; q)$ is transformed into the source $(a^*; -q)$, where a^* is the point symmetrical to a. The function $u = \dfrac{q}{2\pi} \log \dfrac{1}{|f(z, a)|} + c$, where $f(z, a)$ conformally maps the domain D onto the unit disk (here and in what follows the coefficient of heat conductivity k is assumed to be equal to 1).

1505. $u = \dfrac{q}{2\pi} \log \left| \dfrac{z - \bar{a}}{z - a} \right| + c$.

1506. $u = \dfrac{q}{2\pi} \log \left| \dfrac{R^2 - \bar{a}z}{R(z - a)} \right| + c$.

1507. $u = \dfrac{q}{2\pi} \log \left| \dfrac{\sin \dfrac{\pi z}{2a} + i \sinh \dfrac{\pi h}{2a}}{\sin \dfrac{\pi z}{2a} - i \sinh \dfrac{\pi h}{2a}} \right| + c$.

1508. $u = \dfrac{q}{2k} \log \left| \dfrac{t + i/\sqrt{k}}{t - i/\sqrt{k}} \right| + c$, $t = \operatorname{sn}\left[\dfrac{K}{a}(z + ib), u\right]$, where k is determined from the relation $\dfrac{K'}{K} = \dfrac{2b}{a}$.

1509. (1) The Green's function $g(z, a)$ of the domain D can be considered as the temperature created in D by the heat source $(a; 2\pi)$ when the temperature on the boundary is equal to zero;

(2) $u(z) = \dfrac{q}{2\pi} g(z, a) + \sum_{k=1}^{n} u_k \omega_k(z)$, where $\omega_k(z)$ is the harmonic measure of Γ_k.

1510. $u = \dfrac{q}{2\pi} g(z, a) + \dfrac{u_2 - u_1}{\log (r_2/r_1)} \log \dfrac{|z|}{r_1} + u_1$, where $g(z, a)$ is the Green function (see the answer to problem **1503**).